

# The fundamental solution of the space-time fractional diffusion equation

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*Dedicated to Rudolf Gorenflo, Professor Emeritus of the Free University of Berlin, on the occasion of his 70-th birthday (July 31, 2000).*

## Abstract

We deal with the Cauchy problem for the space-time fractional diffusion equation, which is obtained from the standard diffusion equation by replacing the second-order space derivative with a Riesz-Feller derivative of order  $\alpha \in (0, 2]$  and skewness  $\theta$  ( $|\theta| \leq \min\{\alpha, 2 - \alpha\}$ ), and the first-order time derivative with a Caputo derivative of order  $\beta \in (0, 2]$ . The fundamental solution (Green function) for the Cauchy problem is investigated with respect to its scaling and similarity properties, starting from its Fourier-Laplace representation. We review the particular cases of space-fractional diffusion  $\{0 < \alpha \leq 2, \beta = 1\}$ , time-fractional diffusion  $\{\alpha = 2, 0 < \beta \leq 2\}$ , and neutral-fractional diffusion  $\{0 < \alpha = \beta \leq 2\}$ , for which the fundamental solution can be interpreted as a *spatial probability density function evolving*

*in time.* Then, by using the Mellin transform, we provide a general representation of the Green functions in terms of Mellin-Barnes integrals in the complex plane, which allows us *to extend the probability interpretation to the ranges*  $\{0 < \alpha \leq 2\} \cap \{0 < \beta \leq 1\}$  *and*  $\{1 < \beta \leq \alpha \leq 2\}$ . Furthermore, from this representation we derive explicit formulae (convergent series and asymptotic expansions), which enable us to plot the spatial probability densities for different values of the relevant parameters  $\alpha, \theta, \beta$ .

*MSC:* 26A33, 33E12, 33C40, 44A10, 45K05, 60G52.

*Key Words:* diffusion, fractional derivative, Fourier transform, Laplace transform, Mellin transform, Mittag-Leffler function, Wright function, Mellin-Barnes integrals, Green function, stable probability distributions.

## 1. Introduction

A space-time fractional diffusion equation, obtained from the standard diffusion equation by replacing the second order space-derivative by a fractional Riesz derivative order  $\alpha > 0$  and the first order time-derivative by a fractional derivative of order  $\beta > 0$  (in Caputo or Riemann-Liouville sense), has been recently treated by a number of authors, see for example Saichev and Zaslavsky [38], Uchaikin and Zolotarev [48], Gorenflo, Iskenderov and Luchko [17], Scalas, Gorenflo and Mainardi [42], Metzler and Klafter [34]. For other treatments of the space fractional and/or time fractional diffusion equations we refer the reader to the references cited therein. See below for the restrictions on  $\alpha$  and  $\beta$ . In this paper we intend to complement the results obtained in [17] by allowing asymmetry in the space fractional derivative. We thus consider the space-time fractional diffusion equation

$${}_x D_\theta^\alpha u(x, t) = {}_t D_*^\beta u(x, t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \quad (1.1)$$

where the  $\alpha, \theta, \beta$  are real parameters *always* restricted as follows

$$0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}, \quad 0 < \beta \leq 2. \quad (1.2)$$

In (1.1)  $u = u(x, t)$  is the (real) field variable,  ${}_x D_\theta^\alpha$  is the *Riesz-Feller* space-fractional derivative of order  $\alpha$  and skewness  $\theta$ , and  ${}_t D_*^\beta$  is the Caputo time-fractional derivative of order  $\beta$ . These fractional derivatives are integro-differential operators to be defined later.

The paper is divided as follows. In Section 2 we provide the reader with the essential notions and notations concerning the Fourier, Laplace and Mellin transforms, which are necessary in the following.

In Section 3 we introduce the Cauchy problem for the equation (1.1) and find the corresponding fundamental solution  $G_{\alpha,\beta}^\theta(x,t)$  (the *Green function*) in terms of its Fourier-Laplace transform from which we derive its general scaling properties and the similarity variable  $x/t^{\beta/\alpha}$ . We shall get the fundamental formula

$$G_{\alpha,\beta}^\theta(x,t) = t^{-\gamma} K_{\alpha,\beta}^\theta(x/t^\gamma), \quad \gamma = \beta/\alpha, \quad (1.3)$$

where  $K_{\alpha,\beta}^\theta(x)$  is referred to as the *reduced Green function*.

In Section 4, we consider the particular cases  $\{0 < \alpha \leq 2, \beta = 1\}$  (*space fractional diffusion*),  $\{\alpha = 2, 0 < \beta \leq 2\}$  (*time fractional diffusion*), and  $\{0 < \alpha = \beta \leq 2\}$  (*neutral fractional diffusion*), for which the fundamental solution can be interpreted as a spatial *probability density function (pdf)*, evolving in time.

In Section 5 we show a composition rule for the Green function, valid for

$$\{0 < \alpha \leq 2\} \cap \{0 < \beta \leq 1\}$$

which ensures its probability interpretation in this range.

In Section 6, we provide a general representation for the (reduced) Green function in terms of a Mellin-Barnes integral in the complex plane, which enables us to extend the probability interpretation to the range

$$\{1 < \beta \leq \alpha \leq 2\}.$$

In Section 7, we derive for the Green function explicit formulae (convergent series and asymptotic expansions), whose form depends on the relation between the parameters  $\alpha$  and  $\beta$ . By means of a suitable matching between the convergent and the asymptotic expansion we shall be able to compute the Green function in all the cases in which it is interpretable as a probability density.

Finally, Section 8 is devoted to concluding discussions, and a summary of the results in which we present plots of the Green function for a number of cases.

## 2. Notions and Notations

For the sake of the reader's convenience here we present an introduction to the *Riesz-Feller* and *Caputo* fractional derivatives starting from their representation in the Fourier and Laplace transform domain, respectively. So doing we avoid the subtleties lying in the inversion of fractional integrals, see *e.g.* [39], [21]. We also recall the main properties of the Mellin transform that will be used later.

Since in what follows we shall meet only real or complex-valued functions of a real variable that are defined and continuous in a given open interval  $\mathcal{I} = (a, b)$ ,  $-\infty \leq a < b \leq +\infty$ , except, possibly, at isolated points where these functions can be infinite, we restrict our presentation of the integral transforms to the class of functions for which the Riemann improper integral on  $\mathcal{I}$  absolutely converges. In so doing we follow Marichev [32] and we denote this class by  $L^c(\mathcal{I})$  or  $L^c(a, b)$ .

*The Fourier transform and the Riesz-Feller space-fractional derivative*

Let

$$\widehat{f}(\kappa) = \mathcal{F}\{f(x); \kappa\} = \int_{-\infty}^{+\infty} e^{+i\kappa x} f(x) dx, \quad \kappa \in \mathbb{R}, \quad (2.1a)$$

be the Fourier transform of a function  $f(x) \in L^c(\mathbb{R})$ , and let

$$f(x) = \mathcal{F}^{-1}\{\widehat{f}(\kappa); x\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\kappa x} \widehat{f}(\kappa) d\kappa, \quad x \in \mathbb{R}, \quad (2.1b)$$

be the inverse Fourier transform<sup>1</sup>. For a sufficiently well-behaved function  $f(x)$  we define the *Riesz-Feller* space-fractional derivative of order  $\alpha$  and skewness  $\theta$  as

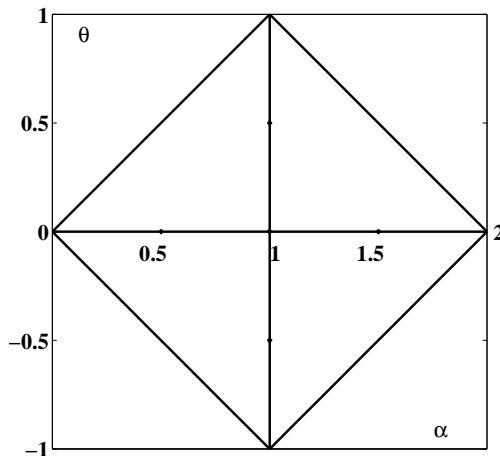
$$\mathcal{F}\{{}_x D_\theta^\alpha f(x); \kappa\} = -\psi_\alpha^\theta(\kappa) \widehat{f}(\kappa), \quad (2.2)$$

$$\psi_\alpha^\theta(\kappa) = |\kappa|^\alpha e^{i(\text{sign } \kappa)\theta\pi/2}, \quad 0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}. \quad (2.3)$$

We note that the allowed region for the parameters  $\alpha$  and  $\theta$  turns out to be a diamond in the plane  $\{\alpha, \theta\}$  with vertices in the points  $(0, 0)$ ,  $(1, 1)$ ,  $(1, -1)$ ,  $(2, 0)$ , that we call the *Feller-Takayasu diamond*, see Fig. 1. Thus, we recognize that the *Riesz-Feller* derivative is required to be

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<sup>1</sup>If  $f(x)$  is piecewise differentiable, then the formula (2.1b) holds true at all points where  $f(x)$  is continuous and the integral in it must be understood in the sense of the Cauchy principal value.



**Fig. 1** The Feller-Takayasu diamond

the pseudo-differential operator<sup>2</sup> whose symbol  $-\psi_{\alpha}^{\theta}(\kappa)$  is the logarithm of the characteristic function of a general *Lévy strictly stable* probability density with *index of stability*  $\alpha$  and asymmetry parameter  $\theta$  (improperly called *skewness*) according to Feller's parameterization [13, 14], as revisited by Gorenflo and Mainardi, see [22, 23, 24].

The operator defined by (2.2)-(2.3) has been referred to as the *Riesz-Feller* fractional derivative since it is obtained as the left inverse of a fractional integral originally introduced (for  $\theta = 0$  and  $\alpha \neq 1$ ) by Marcel Riesz in the late 1940's, known as the *Riesz potential*, and then generalized (for  $\theta \neq 0$ ) by William Feller in 1952, see [13], [39].

For more details on Lévy stable densities we refer the reader to specialistic treatises, as Feller [14], Zolotarev [49], Samorodnitsky and Taqqu [40], Janicki and Weron [25], Sato[41], Uchaikin and Zolotarev[48], where different notations are adopted. We like to refer also to the 1986 paper by Schneider [43], where he first provided the Fox *H*-function representation of the stable distributions (with  $\alpha \neq 1$ ) and to the 1990 book by Takayasu [45] where he first gave the diamond representation in the plane  $\{\alpha, \theta\}$ .

For  $\theta = 0$  we have a symmetric operator with respect to  $x$ , which can be

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<sup>2</sup>Let us recall that a generic pseudo-differential operator  $A$ , acting with respect to the variable  $x \in \mathbb{R}$ , is defined through its Fourier representation, namely  $\int_{-\infty}^{+\infty} e^{i\kappa x} A[f(x)] dx = \widehat{A}(\kappa) \widehat{f}(\kappa)$ , where  $\widehat{A}(\kappa)$  is referred to as symbol of  $A$ , given as  $\widehat{A}(\kappa) = (A e^{-i\kappa x}) e^{+i\kappa x}$ .

interpreted as

$${}_x D_0^\alpha = - \left( -\frac{d^2}{dx^2} \right)^{\alpha/2}, \quad (2.4)$$

as can be formally deduced by writing  $-|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}$ . We thus recognize that the operator  $D_0^\alpha$  is related to a power of the positive definite operator  $-{}_x D^2 = -\frac{d^2}{dx^2}$  and must not be confused with a power of the first order differential operator  ${}_x D = \frac{d}{dx}$  for which the symbol is  $-i\kappa$ . An alternative illuminating notation for the symmetric fractional derivative is due to Zaslavsky, see *e.g.* [38], and reads

$${}_x D_0^\alpha = \frac{d^\alpha}{d|x|^\alpha}. \quad (2.5)$$

In its regularized form valid for  $0 < \alpha < 2$  the *Riesz* space-fractional derivative admits the explicit representation<sup>3</sup>

$${}_x D_0^\alpha f(x) = \frac{\Gamma(1+\alpha)}{\pi} \sin\left(\frac{\alpha\pi}{2}\right) \int_0^\infty \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{1+\alpha}} d\xi. \quad (2.6)$$

For  $\alpha = 1$  the Riesz derivative is related to the Hilbert transform as first noted by Feller in 1952 in his pioneering paper [13], resulting in the formula

$${}_x D_0^1 f(x) = -\frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{f(\xi)}{x-\xi} d\xi. \quad (2.7)$$

For  $0 < \alpha < 2$  and  $|\theta| \leq \min\{\alpha, 2-\alpha\}$  the *Riesz-Feller* derivative reads

$$\begin{aligned} {}_x D_\theta^\alpha f(x) = \frac{\Gamma(1+\alpha)}{\pi} \left\{ \sin[(\alpha+\theta)\pi/2] \int_0^\infty \frac{f(x+\xi) - f(x)}{\xi^{1+\alpha}} d\xi \right. \\ \left. + \sin[(\alpha-\theta)\pi/2] \int_0^\infty \frac{f(x-\xi) - f(x)}{\xi^{1+\alpha}} d\xi \right\}. \end{aligned} \quad (2.8)$$

For  $\alpha = 1$  we obtain the composite formula

$${}_x D_\theta^1 f(x) = \left[ \cos(\theta\pi/2) {}_x D_0^1 + \sin(\theta\pi/2) {}_x D \right] f(x), \quad (2.9)$$

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<sup>3</sup>The representation (2.6), based on a suitable regularization of a hyper-singular integral, can be found in Samko, Kilbas & Marichev [39] as formula (12.1') and is more explicit and convenient than other ones available in the literature, see *e.g.* Saichev & Zaslavsky [38], Uchaikin & Zolotarev [48], in that it is valid in the whole range  $0 < \alpha < 2$ . Gorenflo and Mainardi have used it in [24], where they have shown that it holds also in the singular case  $\alpha = 1$ .

that in the extremal cases  $\theta = \pm 1$  reduces to

$${}_x D_{\pm 1}^1 = \pm {}_x D = \pm \frac{d}{dx}. \quad (2.10)$$

*The Laplace transform and the Caputo fractional derivative*

Now we present an introduction to the *Caputo* fractional derivative starting from its representation in the Laplace transform domain and contrasting it to the standard *Riemann-Liouville* fractional derivative.

Let

$$\tilde{f}(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt, \quad \Re(s) > a_f, \quad (2.11a)$$

be the Laplace transform of a function  $f(t) \in \mathcal{L}^c(0, T)$ ,  $\forall T > 0$  and let

$$f(t) = \mathcal{L}^{-1}\{\tilde{f}(s); t\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \tilde{f}(s) ds, \quad \Re(s) = \gamma > a_f, \quad (2.11b)$$

with  $t > 0$ , be the inverse Laplace transform<sup>4</sup> For a sufficiently well-behaved function  $f(t)$  we define the *Caputo* time-fractional derivative of order  $\beta$  ( $m - 1 < \beta \leq m$ ,  $m \in \mathbb{N}$ ) through

$$\mathcal{L}\{{}_t D_*^\beta f(t); s\} = s^\beta \tilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-1-k} f^{(k)}(0^+), \quad m - 1 < \beta \leq m. \quad (2.12)$$

This leads to define, see *e.g.* [5], [21],

$${}_t D_*^\beta f(t) := \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau) d\tau}{(t-\tau)^{\beta+1-m}}, & m - 1 < \beta < m, \\ \frac{d^m}{dt^m} f(t), & \beta = m. \end{cases} \quad (2.13)$$

The operator defined by (2.12)-(2.13) has been referred to as the *Caputo* fractional derivative since it was introduced by Caputo in the late 1960's for

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<sup>4</sup>A sufficient condition of the existence of the Laplace transform is that the original function is of exponential order as  $t \rightarrow \infty$ . This means that some constant  $a_f$  exists such that the product  $e^{-a_f t} |f(t)|$  is bounded for all  $t$  greater than some  $T$ . Then  $\tilde{f}(s)$  exists and is analytic in the half plane  $\Re(s) > a_f$ . If  $f(t)$  is piecewise differentiable, then the formula (2.11b) holds true at all points where  $f(t)$  is continuous and the (complex) integral in it must be understood in the sense of the Cauchy principal value.

modelling the energy dissipation in some anelastic materials with memory, see [4, 5]. A former review of the theoretical aspects of this derivative with applications in viscoelasticity was given in 1971 by Caputo and Mainardi [8], with special emphasis to the long-memory effects.

The reader should observe that the *Caputo* fractional derivative differs from the usual *Riemann-Liouville* fractional derivative which, defined as the left inverse of the Riemann-Liouville fractional integral, is here denoted as  ${}_tD^\beta f(t)$ . We have, see *e.g.* [39],

$${}_tD^\beta f(t) := \begin{cases} \frac{d^m}{dt^m} \left[ \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{\beta+1-m}} \right], & m-1 < \beta < m, \\ \frac{d^m}{dt^m} f(t), & \beta = m. \end{cases} \quad (2.14)$$

When the order is not integer, Gorenflo and Mainardi have shown the following relationships between the two fractional derivatives (when both of them exist), see *e.g.* [21],

$${}_tD_*^\beta f(t) = {}_tD^\beta \left[ f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!} \right], \quad m-1 < \beta < m, \quad (2.15)$$

or

$${}_tD_*^\beta f(t) = {}_tD^\beta f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^{k-\beta}}{\Gamma(k-\beta+1)}, \quad m-1 < \beta < m. \quad (2.16)$$

The *Caputo* fractional derivative is of course more restrictive than the *Riemann-Liouville* fractional derivative in that the derivative of order  $m$  is required to exist and be absolutely Laplace transformable. Whenever we use the operator  ${}_tD_*^\beta$ , we (tacitly) assume that this condition is met.

The *Caputo* fractional derivative<sup>5</sup>, practically ignored in the mathematical treatises, represents a sort of regularization in the time origin for the *Riemann-Liouville* fractional derivative. Recently, it has been extensively investigated by Gorenflo & Mainardi [21] and by Podlubny [36] in view of its major utility in treating physical and engineering problems which require standard initial conditions. Several applications have been treated by Caputo himself up to nowadays, see *e.g.* [6, 7] and references therein.

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<sup>5</sup>According to Samko, Kilbas & Marichev[39] and Butzer & Westphal[3] the "regularized" fractional derivative was considered by Liouville himself (but then disregarded).



We point out that the *Caputo* fractional derivative satisfies the relevant property of being zero when applied to a constant, and, in general, to any power function of non-negative integer degree less than  $m$ , if its order  $\beta$  is such that  $m - 1 < \beta < m$ . Furthermore, since

$${}_t D^\beta t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \beta)} t^{\gamma - \beta}, \quad \beta > 0, \quad \gamma > -1, \quad t > 0, \quad (2.17)$$

we note that

$${}_t D^\beta f(t) = {}_t D^\beta g(t) \iff f(t) = g(t) + \sum_{j=1}^m c_j t^{\beta - j}, \quad (2.18)$$

whereas, using also (2.15) or (2.16),

$${}_t D_*^\beta f(t) = {}_t D_*^\beta g(t) \iff f(t) = g(t) + \sum_{j=1}^m c_j t^{m - j}. \quad (2.19)$$

In these formulae the coefficients  $c_j$  are arbitrary constants. We also note the different behaviour of  ${}_t D_*^\beta$  at the end points of the interval  $(m - 1, m)$ ,

$$\lim_{\beta \rightarrow (m-1)^+} {}_t D_*^\beta f(t) = f^{(m-1)}(t) - f^{(m-1)}(0^+), \quad \lim_{\beta \rightarrow m^-} {}_t D_*^\beta f(t) = f^{(m)}(t). \quad (2.20)$$

The last limit can be formally obtained by recalling the formal representation of the  $m$ -th derivative of the Dirac function,  $\delta^{(m)}(t) = t^{-m-1}/\Gamma(-m)$ ,  $t \geq 0$ , see [16]. As a consequence of (2.20), with respect to the order, the *Caputo* derivative is an operator left-continuous at any positive integer.

### *The Mellin transform*

Let

$$\mathcal{M}\{f(r); s\} = f^*(s) = \int_0^{+\infty} f(r) r^{s-1} dr, \quad \gamma_1 < \Re(s) < \gamma_2 \quad (2.21a)$$

be the Mellin transform of a sufficiently well-behaved function  $f(r)$ , and let

$$\mathcal{M}^{-1}\{f^*(s); r\} = f(r) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s) r^{-s} ds \quad (2.21b)$$

be the inverse Mellin transform<sup>6</sup>, where  $r > 0$ ,  $\gamma = \Re(s)$ ,  $\gamma_1 < \gamma < \gamma_2$ .

We refer to specialized treatises and/or handbooks, see *e.g.* [12], [32], [37], for more details and tables on the Mellin transform. Here, for our convenience we recall the main rules that are useful to adapt the formulae from the handbooks and, meantime, are relevant in the following.

Denoting by  $\overset{\mathcal{M}}{\leftrightarrow}$  the juxtaposition of a function  $f(r)$  with its Mellin transform  $f^*(s)$ , the main rules are:

$$f(ar) \overset{\mathcal{M}}{\leftrightarrow} a^{-s} f^*(s), \quad a > 0, \quad (2.22)$$

$$r^a f(r) \overset{\mathcal{M}}{\leftrightarrow} f^*(s+a), \quad (2.23)$$

$$f(r^p) \overset{\mathcal{M}}{\leftrightarrow} \frac{1}{|p|} f^*(s/p), \quad p \neq 0, \quad (2.24)$$

$$h(r) = \int_0^\infty \frac{1}{\rho} f(\rho) g(r/\rho) d\rho \overset{\mathcal{M}}{\leftrightarrow} h^*(s) = f^*(s) g^*(s). \quad (2.25)$$

The Mellin convolution formula (2.25) is useful in treating integrals of Fourier type for  $x = |x| > 0$  :

$$I_c(x) = \frac{1}{\pi} \int_0^\infty f(\kappa) \cos(\kappa x) d\kappa, \quad (2.26)$$

$$I_s(x) = \frac{1}{\pi} \int_0^\infty f(\kappa) \sin(\kappa x) d\kappa, \quad (2.27)$$

when the Mellin transform  $f^*(s)$  of  $f(\kappa)$  is known. In fact we recognize that the integrals  $I_c(x)$  and  $I_s(x)$  can be interpreted as Mellin convolutions

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<sup>6</sup>For the existence of the Mellin transform and the validity of the inversion formula we need to recall the following theorems, <sup>(6a)</sup> and <sup>(6b)</sup>, adapted from Marichev's treatise [32], see THEOREMS 11, 12, at page 39.

<sup>(6a)</sup> Let  $f(r) \in L^c(\epsilon, E)$ ,  $0 < \epsilon < E < \infty$ , be continuous in the intervals  $(0, \epsilon]$ ,  $[E, \infty)$ , and let  $|f(r)| \leq M r^{-\gamma_1}$  for  $0 < r < \epsilon$ ,  $|f(r)| \leq M r^{-\gamma_2}$  for  $r > E$ , where  $M$  is a constant. Then for the existence of a strip in the  $s$ -plane in which  $f(r) r^{s-1}$  belongs to  $L^c(0, \infty)$  it is sufficient that  $\gamma_1 < \gamma_2$ . When this condition holds, the Mellin transform  $f^*(s)$  exists and is analytic in the vertical strip  $\gamma_1 < \gamma = \Re(s) < \gamma_2$ .

<sup>(6b)</sup> If  $f(t)$  is piecewise differentiable, and  $f(r) r^{\gamma-1} \in L^c(0, \infty)$ , then the formula (2.21b) holds true at all points where  $f(r)$  is continuous and the (complex) integral in it must be understood in the sense of the Cauchy principal value.

(2.25) between  $f(\kappa)$  and the functions  $g_c(\kappa)$ ,  $g_s(\kappa)$ , respectively, with  $r = 1/|x|$ ,  $\rho = \kappa$ , where

$$g_c(\kappa) := \frac{1}{\pi |x| \kappa} \cos\left(\frac{1}{\kappa}\right) \stackrel{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(1-s)}{\pi |x|} \sin\left(\frac{\pi s}{2}\right) := g_c^*(s), \quad 0 < \Re(s) < 1, \quad (2.28)$$

$$g_s(\kappa) := \frac{1}{\pi |x| \kappa} \sin\left(\frac{1}{\kappa}\right) \stackrel{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(1-s)}{\pi |x|} \cos\left(\frac{\pi s}{2}\right) := g_s^*(s), \quad 0 < \Re(s) < 2. \quad (2.29)$$

The Mellin transform pairs (2.28)-(2.29) have been adapted from the tables in [32] by using (2.22)-(2.24) and the duplication and reflection formulae for the Gamma function. Finally, the inverse Mellin transform representation (2.21b) provides the required integrals as

$$I_c(x) = \frac{1}{\pi x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) x^s ds, \quad x > 0, \quad 0 < \gamma < 1, \quad (2.30)$$

$$I_s(x) = \frac{1}{\pi x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) \Gamma(1-s) \cos\left(\frac{\pi s}{2}\right) x^s ds, \quad x > 0, \quad 0 < \gamma < 2. \quad (2.31)$$

### 3. Scaling and similarity properties of the Green function

For the equation (1.1) we consider the Cauchy problem

$$u(x, 0) = \varphi(x), \quad x \in \mathbb{R}, \quad u(\pm\infty, t) = 0, \quad t > 0, \quad (3.1)$$

where  $\varphi(x) \in L^c(\mathbb{R})$  is a sufficiently well-behaved function. If  $1 < \beta \leq 2$  we add to (3.1) the condition  $u_t(x, 0) = 0$ , where  $u_t(x, t) = \frac{\partial}{\partial t} u(x, t)$ .

By solution of the Cauchy problem for the equation (1.1) we mean a function  $u_{\alpha, \beta}^\theta(x, t)$  which satisfies the conditions (3.1). By the Green function (or fundamental solution) of the Cauchy problem we mean the (generalized) function  $G_{\alpha, \beta}^\theta(x, t)$  which, being the formal solution of (1.1) corresponding to  $\varphi(x) = \delta(x)$  (the Dirac delta function), allows us to represent the solution of the Cauchy problem by the integral formula

$$u_{\alpha, \beta}^\theta(x, t) = \int_{-\infty}^{+\infty} G_{\alpha, \beta}^\theta(\xi, t) \varphi(x - \xi) d\xi. \quad (3.2)$$

It is straightforward to derive from (1.1) and (3.1) the Fourier-Laplace transform of the Green function by taking into account the Fourier transform for the *Riesz-Feller* space-fractional derivative, see (2.2), and the Laplace transform for the *Caputo* time-fractional derivative, see (2.12). We have (in an obvious notation):

$$-\psi_\alpha^\theta(\kappa) \widehat{G}_{\alpha,\beta}^\theta(\kappa, s) = s^\beta \widehat{G}_{\alpha,\beta}^\theta(\kappa, s) - s^{\beta-1}, \quad (3.3)$$

where

$$\psi_\alpha^\theta(\kappa) := |\kappa|^\alpha e^{i(\text{sign } \kappa)\theta\pi/2} = \overline{\psi_\alpha^\theta(-\kappa)} = \psi_\alpha^{-\theta}(-\kappa). \quad (3.4)$$

We therefore obtain

$$\widehat{G}_{\alpha,\beta}^\theta(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + \psi_\alpha^\theta(\kappa)}. \quad (3.5)$$

By using the known scaling rules for the Fourier and Laplace transforms,

$$f(ax) \xleftrightarrow{\mathcal{F}} a^{-1} \widehat{f}(\kappa/a), \quad a > 0, \quad f(bt) \xleftrightarrow{\mathcal{L}} b^{-1} \widetilde{f}(s/b), \quad b > 0, \quad (3.6)$$

we infer directly from (3.5) (thus without inverting the two transforms) the following *scaling property* of the Green function,

$$G_{\alpha,\beta}^\theta(ax, bt) = b^{-\gamma} G_{\alpha,\beta}^\theta(ax/b^\gamma, t), \quad \gamma = \beta/\alpha. \quad (3.7)$$

Consequently, introducing the *similarity variable*  $x/t^\gamma$ , we can write

$$G_{\alpha,\beta}^\theta(x, t) = t^{-\gamma} K_{\alpha,\beta}^\theta(x/t^\gamma), \quad \gamma = \beta/\alpha, \quad (3.8)$$

where the one-variable function  $K_{\alpha,\beta}^\theta$  is to be determined as indicated below.

Let us first invert the Laplace transform in (3.5). To this purpose we recall the Laplace transform pair,

$$E_\beta(ct^\beta) \xleftrightarrow{\mathcal{L}} \frac{s^{\beta-1}}{s^\beta - c}, \quad \Re(s) > |c|^{1/\beta}, \quad (3.9)$$

with  $c \in \mathcal{C}$ ,  $0 < \beta \leq 2$ , where  $E_\beta$  denotes the entire transcendental function, known as the Mittag-Leffler function of order  $\beta$ , defined in the complex plane by the power series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad \beta > 0, \quad z \in \mathcal{C}. \quad (3.10)$$

Then, by comparing (3.5) with (3.9), we obtain the Fourier transform of the Green function as

$$\widehat{G}_{\alpha,\beta}^{\theta}(\kappa, t) = E_{\beta} \left[ -\psi_{\alpha}^{\theta}(\kappa) t^{\beta} \right], \quad \kappa \in \mathbb{R}, \quad t \geq 0. \quad (3.11)$$

For detailed information on the Mittag-Leffler-type functions and their Laplace transforms the reader may consult *e.g.* [11], [10], [39], [26], [20], [21], [36]. We note for later use that the Mittag-Leffler function (3.10) admits a Mellin transform type representation, originally due to Barnes [1] (see also [32] p. 118 (7.79)), as

$$E_{\beta}(z) = \frac{1}{2\pi i} \int_{L_{-\infty}} \frac{\Gamma(s) \Gamma(1-s)}{\Gamma(1-\beta s)} (-z)^{-s} ds, \quad (3.12)$$

where the integration is over a left-hand loop  $L_{-\infty}$  drawn round all the left-hand poles  $s = 0, -1, -2, \dots$  of the integrand in a positive direction. Indeed, by the residue theorem it is not difficult to write the integral in (3.12) as the power series in (3.10). As a matter of fact, from (3.12) we can deduce the following Mellin transform pair (see also [32], p. 300) which will be used to get the Fourier antitransform of (3.11),

$$E_{\beta}(-r) \stackrel{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(s) \Gamma(1-s)}{\Gamma(1-\beta s)}, \quad r > 0, \quad 0 < \beta \leq 2, \quad 0 < \Re(s) < 1. \quad (3.13)$$

In view of the symmetry properties of  $\psi_{\alpha}^{\theta}(\kappa)$ , see (3.4), and of the Mittag-Leffler function *i.e.*  $E_{\beta}(z) = \overline{E_{\beta}(\bar{z})}$ ,  $z \in \mathcal{C}$ , we have

$$\widehat{G}_{\alpha,\beta}^{\theta}(\kappa, t) = \overline{\widehat{G}_{\alpha,\beta}^{\theta}(-\kappa, t)} = \widehat{G}_{\alpha,\beta}^{-\theta}(-\kappa, t). \quad (3.14)$$

Furthermore, we easily recognize from (3.10)-(3.11),

$$\widehat{G}_{\alpha,\beta}^{\theta}(0, t) = E_{\beta}(0) = 1, \quad t \geq 0. \quad (3.15)$$

Provided that  $G_{\alpha,\beta}^{\theta}(x, t)$  does exist as inverse Fourier transform of (3.11), equations (3.14)-(3.15) ensure that  $G_{\alpha,\beta}^{\theta}(x, t)$  is real and normalized, *i.e.*  $\int_{-\infty}^{+\infty} G_{\alpha,\beta}^{\theta}(x, t) dx = 1$ .

The inversion of the Fourier transform (at most as a improper integral) requires (in view of the Riemann-Lebesgue lemma) that for  $t \geq 0$ ,

$$\left| E_{\beta} \left[ -\psi_{\alpha}^{\theta}(\kappa) t^{\beta} \right] \right| \rightarrow 0 \quad \text{as} \quad |\kappa| \rightarrow \infty. \quad (3.16)$$

Taking into account the growth properties of the Mittag-Leffler function, see *e.g.* Gorenflo, Luchko and Rogosin [20], this means that the LHS of (3.16) is required to be bounded for  $\kappa \in \mathbb{R}$  and  $t \geq 0$ . It turns out that for this we must require

$$|\theta| \leq 2 - \beta. \quad (3.17)$$

This means that for  $1 < \beta < 2$  the allowed  $\{\alpha, \theta\}$  region could be no longer the *Feller-Takayasu diamond*  $|\theta| \leq \min\{\alpha, 2 - \alpha\}$ ; in fact, in the cases  $0 < \alpha < 1 < \beta < 2$  with  $\alpha + \beta > 2$  and  $1 < \alpha < \beta < 2$  we must cut-off the upper and lower corners of the diamond with the lines  $\theta = \pm(2 - \beta)$ .

As far as the determination of the one-variable function  $K_{\alpha,\beta}^\theta(x)$  is concerned, see (3.8), we note the *symmetry relation*

$$K_{\alpha,\beta}^\theta(-x) = K_{\alpha,\beta}^{-\theta}(x). \quad (3.18)$$

As a consequence, we can restrict our attention to  $x > 0$ , and obtain

$$K_{\alpha,\beta}^\theta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\kappa x} E_\beta[-\psi_\alpha^\theta(\kappa)] d\kappa = {}_cK_{\alpha,\beta}^\theta(x) + {}_sK_{\alpha,\beta}^\theta(x), \quad (3.19)$$

$${}_cK_{\alpha,\beta}^\theta(x) = \frac{1}{\pi} \int_0^\infty \cos(\kappa x) \Re[E_\beta(-\kappa^\alpha e^{i\theta\pi/2})] d\kappa, \quad (3.20)$$

$${}_sK_{\alpha,\beta}^\theta(x) = \frac{1}{\pi} \int_0^\infty \sin(\kappa x) \Im[E_\beta(-\kappa^\alpha e^{i\theta\pi/2})] d\kappa. \quad (3.21)$$

From (3.19) we can easily obtain the value attained by  $K_{\alpha,\beta}^\theta(x)$  in  $x = 0$ . To this purpose we extend the argument in [17] writing

$$\begin{aligned} K_{\alpha,\beta}^\theta(0) &= \frac{1}{\pi} \Re \left[ \int_0^\infty E_\beta(-\kappa^\alpha e^{i\theta\pi/2}) d\kappa \right] \\ &= \cos\left(\frac{\theta\pi}{2\alpha}\right) \frac{1}{\pi\alpha} \int_0^\infty E_\beta(-r) r^{1/\alpha} - 1 dr. \end{aligned} \quad (3.22)$$

Thus, the last integral above is the Mellin transform of the Mittag-Leffler function at the point  $s = 1/\alpha$ ; it turns out to be convergent under the conditions  $\beta = 1$  with  $\alpha > 0$ , and  $0 < \beta \leq 2$ ,  $\beta \neq 1$  with  $\alpha > 1$ . For the finite value of  $K_{\alpha,\beta}^\theta(x)$  at  $x = 0$  we thus use (3.13) obtaining

$$K_{\alpha,\beta}^\theta(0) = \begin{cases} \frac{1}{\pi\alpha} \frac{\Gamma(1/\alpha)\Gamma(1-1/\alpha)}{\Gamma(1-\beta/\alpha)} \cos\left(\frac{\theta\pi}{2\alpha}\right), & 1 < \alpha \leq 2, \beta \neq 1, \\ \frac{1}{\pi\alpha} \Gamma(1/\alpha) \cos\left(\frac{\theta\pi}{2\alpha}\right), & 0 < \alpha \leq 2, \beta = 1. \end{cases} \quad (3.23)$$

We note that  $K_{\alpha,\beta}^\theta(0)$  is non negative except for  $1 < \alpha < \beta \leq 2$ .

## 4. Particular cases for the Green function

In this Section we are going to consider the important particular cases of our space-time fractional diffusion equation, *i.e.*

- $\{\alpha = 2, \beta = 1\}$  (*standard diffusion*),
- $\{0 < \alpha \leq 2, \beta = 1\}$  (*space fractional diffusion*),
- $\{\alpha = 2, 0 < \beta \leq 2, \beta \neq 1\}$  (*time fractional diffusion*),
- $\{0 < \alpha = \beta \leq 2\}$  (*neutral fractional diffusion*),

It is well known that for the *standard diffusion*  $\{\alpha = 2, \beta = 1\}$  the Green function is the Gaussian *pdf*

$$G_{2,1}^0(x, t) = t^{-1/2} \frac{1}{2\sqrt{\pi}} \exp[-x^2/(4t)], \quad -\infty < x < +\infty, \quad t \geq 0, \quad (4.1)$$

with similarity variable  $x/t^{1/2}$ , that evolves in time with moments (of even order)

$$\mu_{2n}(t) := \int_{-\infty}^{+\infty} x^{2n} G_{2,1}^0(x, t) dx = \frac{(2n)!}{n!} t^n, \quad n = 0, 1, 2, \dots, \quad t \geq 0. \quad (4.2)$$

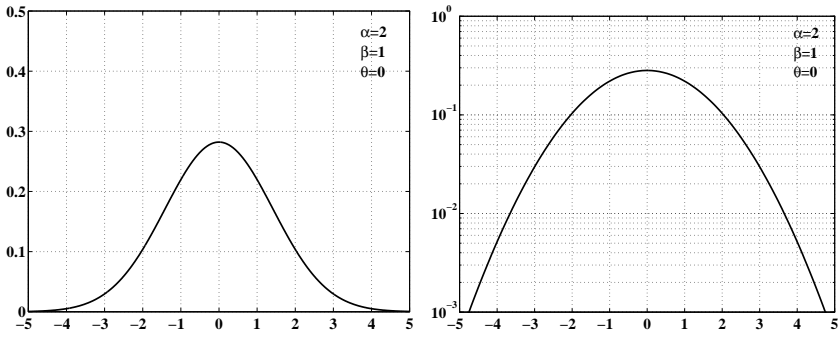
The variance  $\sigma^2 := \mu_2(t) = 2t$  is thus proportional to the first power of time, according to the Einstein diffusion law.

In Fig. 2 we report the plots of the Gaussian.  $K_{2,1}^0(x) = 1/(2\sqrt{\pi}) \exp(-x^2/4)$  in the interval  $-5 \leq x \leq 5$  by adopting for the ordinate a linear scale (left) and a logarithmic scale (right). We recognize that the logarithmic scale is to be preferred to point out the tails. At the end of this paper we shall exhibit a number of lin-log plots of the fundamental solution for different values of the parameters  $\alpha, \theta, \beta$ , that the reader can compare with the corresponding plot of the Gaussian, and emphasize the different behaviour of the tails.

### The space-fractional diffusion

Let us now consider  $\{0 < \alpha \leq 2, \beta = 1\}$  (*space fractional diffusion* including standard diffusion for  $\alpha = 2$ ). In this case, reducing the Mittag-Leffler function in (3.11) to the exponential function, we recover the characteristic function of the class of Lévy strictly stable densities according to the Feller parameterization, see (2.2), (2.3) and (3.4). In fact, denoting this class by  $\{L_\alpha^\theta(x)\}$ , we have

$$\widehat{L}_\alpha^\theta(\kappa) = e^{-\psi_\alpha^\theta(\kappa)}, \quad \text{and} \quad \widehat{G}_{\alpha,1}^\theta(\kappa, t) = e^{-t\psi_\alpha^\theta(\kappa)}, \quad (4.3)$$



**Fig. 2** The Gaussian *pdf*.

with  $\psi_\alpha^\theta(\kappa)$  given by (3.4) and  $\alpha$  and  $\theta$  restricted in the Feller-Takayasu diamond, see (2.3). Then the Green function of the space-fractional diffusion equation can be interpreted as a Lévy strictly stable *pdf*, evolving in time, according to

$$G_{\alpha,1}^\theta(x,t) = t^{-1/\alpha} L_\alpha^\theta(x/t^{1/\alpha}), \quad -\infty < x < +\infty, \quad t \geq 0. \quad (4.4)$$

A stable *pdf* with extremal value of the skewness parameter is called *extremal*. One can prove that all the extremal stable *pdfs*' with  $0 < \alpha < 1$  are one-sided, the support being  $\mathbb{R}_0^+$  if  $\theta = -\alpha$ , and  $\mathbb{R}_0^-$  if  $\theta = +\alpha$ . The one-sided stable *pdf*'s with support in  $\mathbb{R}_0^+$  can be better characterized by their (spatial) Laplace transforms, which turn out to be

$$\widetilde{L}_\alpha^{-\alpha}(s) := \int_0^\infty e^{-sx} L_\alpha^{-\alpha}(x) dx = e^{-s^\alpha}, \quad \Re(s) > 0, \quad 0 < \alpha < 1. \quad (4.5)$$

The stable densities admit a representation in terms of elementary functions only in the following particular cases

$\alpha = 2, \quad \theta = 0, \quad \text{Gauss} :$

$$e^{-\kappa^2} \stackrel{\mathcal{F}}{\Leftrightarrow} L_2^0(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}, \quad -\infty < x < +\infty; \quad (4.6)$$

$\alpha = 1/2, \quad \theta = -1/2, \quad \text{Lévy-Smirnov} :$

$$e^{-s^{1/2}} \stackrel{\mathcal{L}}{\Leftrightarrow} L_{1/2}^{-1/2}(x) = \frac{x^{-3/2}}{2\sqrt{\pi}} e^{-1/(4x)}, \quad x \geq 0; \quad (4.7)$$

$\alpha = 1, \quad \theta = 0, \quad \text{Cauchy} :$

$$e^{-|\kappa|} \stackrel{\mathcal{F}}{\Leftrightarrow} L_1^0(x) = \frac{1}{\pi} \frac{1}{x^2 + 1}, \quad -\infty < x < +\infty. \quad (4.8)$$



We note that the case  $\alpha = 1$  can be easily treated also for  $\theta \neq 0$  taking into account elementary properties of the Fourier transform; we have  $\alpha = 1$ ,  $0 < |\theta| < 1$  :

$$L_1^\theta(x) = \frac{1}{\pi} \frac{\cos(\theta\pi/2)}{[x + \sin(\theta\pi/2)]^2 + [\cos(\theta\pi/2)]^2}, \quad -\infty < x < +\infty; \quad (4.9)$$

$\alpha = 1$ ,  $\theta = \pm 1$  :

$$L_1^{\pm 1}(x) = \delta(x \pm 1), \quad -\infty < x < +\infty. \quad (4.10)$$

We note that in the above singular cases  $\alpha = 1$ ,  $\theta = \pm 1$ , the corresponding Riesz-Feller derivatives  ${}_x D_{\pm 1}^1$  reduce to  $\pm {}_x D$ , see (2.10), so our fractional diffusion equation (1.1) degenerates into kinematic (*i.e.* first-order) wave equations and the corresponding Green functions are  $G_{1,1}^{\pm 1}(x,t) = \delta(x \pm t)$ , meaning pure drift.

For  $0 < \alpha < 2$  the stable *pdf*'s exhibit *fat tails* in such a way that their absolute moment of order  $\nu$  is finite only if  $-1 < \nu < \alpha$ . In fact one can show that for non-Gaussian, not extremal, stable densities the asymptotic decay of the tails is

$$L_\alpha^\theta(x) = O(|x|^{-(\alpha+1)}), \quad x \rightarrow \pm\infty. \quad (4.11)$$

For the extremal densities with  $\alpha \neq 1$  this is valid only for one tail, the other being of exponential order. For  $0 < \alpha < 1$  we have one-sided *pdf*'s: for  $\theta = -\alpha$  the support is  $\mathbb{R}^+$  and the *pdf* tends exponentially to zero as  $x \rightarrow 0^+$ ; for  $\theta = +\alpha$  the support is  $\mathbb{R}^-$  and the *pdf* tends exponentially to zero as  $x \rightarrow 0^-$ . For  $1 < \alpha < 2$  the extremal *pdf*'s are two-sided and exhibit an exponential left tail (as  $x \rightarrow -\infty$ ) if  $\theta = +(2-\alpha)$ , or an exponential right tail (as  $x \rightarrow +\infty$ ) if  $\theta = -(2-\alpha)$ . Consequently, the Gaussian distribution is the unique stable distribution with finite variance. Furthermore, when  $0 < \alpha \leq 1$ , the first absolute moment is infinite so we should use the median instead of the non-existent expected value. However, there is a fundamental property shared by all the stable *pdf*'s that we like to point out: for any  $\alpha$  the corresponding stable *pdf* is *unimodal* and indeed *bell-shaped*, *i.e.* its  $n$ -th derivative has exactly  $n$  zeros, see Gawronski [15].

A general representation of all stable *pdf*'s in terms of higher transcendental functions has been achieved only in 1986; it was Schneider [43], who first has proved that the stable *pdf*'s can be expressed by means of Fox  $H$ -functions, see also Uchaikin & Zolotarev [48].

We note that already in 1952 Feller [13] had obtained, by inverting the Fourier transform of the characteristic function, representations of the stable *pdf*'s in terms of convergent and asymptotic power series. Feller's results have been revisited (and corrected) by Schneider and can be summarized as follows. We restrict our attention to  $x > 0$ , since the evaluations for  $x < 0$  can be obtained using the *symmetry relation*  $L_\alpha^\theta(-x) = L_\alpha^{-\theta}(x)$ .

The convergent expansions are

$$0 < \alpha < 1, \quad |\theta| \leq \alpha :$$

$$L_\alpha^\theta(x) = \frac{1}{\pi x} \sum_{n=1}^{\infty} (-x^{-\alpha})^n \frac{\Gamma(1+n\alpha)}{n!} \sin \left[ \frac{n\pi}{2}(\theta - \alpha) \right], \quad x > 0; \quad (4.12)$$

$$1 < \alpha \leq 2, \quad |\theta| \leq 2 - \alpha$$

$$L_\alpha^\theta(x) = \frac{1}{\pi x} \sum_{n=1}^{\infty} (-x)^n \frac{\Gamma(1+n/\alpha)}{n!} \sin \left[ \frac{n\pi}{2\alpha}(\theta - \alpha) \right], \quad x > 0. \quad (4.13)$$

From the series in (4.12) and the symmetry relation we note that the extremal stable distributions for  $0 < \alpha < 1$  are unilateral, that is vanishing for  $x > 0$  if  $\theta = \alpha$ , vanishing for  $x < 0$  if  $\theta = -\alpha$ .

The asymptotic representations are given by

$$0 < \alpha < 1, \quad -\alpha < \theta \leq \alpha :$$

$$L_\alpha^\theta(x) \sim \frac{1}{\pi x} \sum_{n=1}^{\infty} (-x)^n \frac{\Gamma(1+n/\alpha)}{n!} \sin \left[ \frac{n\pi}{2\alpha}(\theta - \alpha) \right], \quad x \rightarrow 0^+, \quad (4.14)$$

$$0 < \alpha < 1, \quad \theta = -\alpha :$$

$$L_\alpha^{-\alpha}(x) \sim A_1 x^{-a_1} e^{-b_1 x^{-c_1}}, \quad x \rightarrow 0^+, \quad A_1 = \left\{ [2\pi(1-\alpha)]^{-1} \alpha^{1/(1-\alpha)} \right\}^{1/2}, \quad (4.15)$$

$$a_1 = \frac{2-\alpha}{2(1-\alpha)} \quad b_1 = (1-\alpha) \alpha^{\alpha/(1-\alpha)}, \quad c_1 = \frac{\alpha}{1-\alpha};$$

$$1 < \alpha < 2, \quad \alpha - 2 < \theta \leq 2 - \alpha :$$

$$L_\alpha^\theta(x) \sim \frac{1}{\pi x} \sum_{n=1}^{\infty} (-x^{-\alpha})^n \frac{\Gamma(1+n\alpha)}{n!} \sin \left[ \frac{n\pi}{2}(\theta - \alpha) \right], \quad x \rightarrow \infty, \quad (4.16)$$

$$1 < \alpha < 2, \quad \theta = \alpha - 2 :$$

$$L_\alpha^{\alpha-2}(x) \sim A_2 x^{a_2} e^{-b_2 x^{c_2}}, \quad x \rightarrow \infty, \quad A_2 = \left[ 2\pi(\alpha-1) \alpha^{1/(\alpha-1)} \right]^{-1/2}, \quad (4.17)$$

$$a_2 = \frac{2 - \alpha}{2(\alpha - 1)}, \quad b_2 = (\alpha - 1) \alpha^{\alpha/(\alpha-1)}, \quad c_2 = \frac{\alpha}{\alpha - 1}.$$

From (4.12), (4.13) and (4.14), (4.16), we thus note that for non extremal densities the roles of convergent and asymptotic power series are interchanging with respect to the cases  $0 < \alpha < 1$  and  $1 < \alpha < 2$ .

As a consequence of the convergence of the series in (4.12)-(4.13) and of the identity  $L_\alpha^\theta(-x) = L_\alpha^{-\theta}(x)$ . we recognize that the stable *pdf*'s with  $1 < \alpha \leq 2$  are *entire* functions whereas the stable *pdf*'s with  $0 < \alpha < 1$  have the form

$$L_\alpha^\theta(x) = \begin{cases} (1/x) \Phi_1(x^{-\alpha}) & \text{for } x > 0, \\ (1/|x|) \Phi_2(|x|^{-\alpha}) & \text{for } x < 0, \end{cases} \quad (4.18)$$

where  $\Phi_1(z)$  and  $\Phi_2(z)$  are *entire* functions. The case  $\alpha = 1$  ( $|\theta| < 1$ ) can be treated in the limit for  $\alpha \rightarrow 1$  of (4.12) and (4.13), where the corresponding series reduce to geometric series in  $1/x$  and  $x$ , respectively, with a finite radius of convergence. The corresponding stable *pdf*'s are no longer represented by entire functions, as can be noted directly from (4.8)-(4.9).

From (4.12) -(4.13) a sort of *reciprocity relationship* between stable *pdf*'s with index  $\alpha$  and  $1/\alpha$  can be derived as noted by Feller [14]. Assuming  $1/2 \leq \alpha < 1$  and  $x > 0$ , we obtain

$$\frac{1}{x^{\alpha+1}} L_{1/\alpha}^\theta(x^{-\alpha}) = L_\alpha^{\theta^*}(x), \quad \theta^* = \alpha(\theta + 1) - 1. \quad (4.19)$$

A quick check shows that  $\theta^*$  falls within the prescribed range,  $|\theta^*| \leq \alpha$ , provided that  $|\theta| \leq 2 - 1/\alpha$ .

At the end of the paper we shall exhibit some plots the fundamental solution of the space-fractional diffusion equation,

$$G_{\alpha,1}^\theta(x, 1) = K_{\alpha,1}^\theta(x) = L_\alpha^\theta(x) \quad 0 < \alpha < 2, \quad (4.20)$$

in the range  $|x| \leq 5$ .

### The time-fractional diffusion

Let us now consider the case  $\{\alpha = 2, 0 < \beta < 2\}$  (*time-fractional diffusion* including standard diffusion for  $\beta = 1$ ) for which (3.11) reduces to

$$\widehat{G_{2,\beta}^0}(\kappa, t) = E_\beta(-\kappa^2 t^\beta), \quad \kappa \in \mathbb{R}, \quad t \geq 0. \quad (4.21)$$

Following Mainardi, see *e.g.* [27, 28], the problem can be treated with the equivalent Laplace transform

$$\widetilde{G}_{2,\beta}^0(x, s) = \frac{1}{2} s^{\beta/2-1} e^{-|x|s^{\beta/2}}, \quad -\infty < x < +\infty, \quad \Re(s) > 0, \quad (4.22)$$

with solution

$$G_{2,\beta}^0(x, t) = \frac{1}{2} t^{-\beta/2} M_{\beta/2}(|x|/t^{\beta/2}), \quad -\infty < x < +\infty, \quad t \geq 0, \quad (4.23)$$

where  $M_{\beta/2}$  denotes the so-called  $M$  function (of the Wright type) of order  $\beta/2$ , whose general properties are briefly discussed below.

The function  $M_\nu(z)$  is defined for any order  $\nu \in (0, 1)$  and  $\forall z \in \mathcal{C}$  by

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma[-\nu n + (1 - \nu)]} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\nu n) \sin(\pi \nu n), \quad (4.24)$$

It is a special case of the Wright function on which the interested reader can inform himself in several books and articles, *e.g.* [11], [26],[18, 19]. It turns out that  $M_\nu(z)$  is an entire function of order  $\rho = 1/(1 - \nu)$ , which provides a generalization of the Gaussian and of the Airy function. In fact we obtain

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp(-z^2/4), \quad M_{1/3}(z) = 3^{2/3} \text{Ai}(z/3^{1/3}). \quad (4.25)$$

For our purposes (time-fractional diffusion equation) it is relevant to consider the function  $M_\nu$  ( $0 < \nu < 1$ ) for positive argument. In the following we shall denote generically by  $r$  a positive variable (it can be  $|x|$ ,  $t$  or the similarity variable  $|x|/t^{\beta/2}$ ) and by  $c$  a positive constant which plays the role of a scaling parameter. We then point out the related Laplace transform pairs, see *e.g.* Mainardi [28] and Gorenflo, Luchko & Mainardi [18],

$$\begin{cases} M_\nu(r/c) \xleftrightarrow{\mathcal{L}} c E_\nu(-cs), & \Re(s) > 0, & \text{(a)} \\ \frac{1}{r^\nu} M_\nu(cr^{-\nu}) \xleftrightarrow{\mathcal{L}} s^{\nu-1} \exp(-cs^\nu), & \Re(s) > 0, & \text{(b)} \\ \frac{c\nu}{r^{\nu+1}} M_\nu(cr^{-\nu}) \xleftrightarrow{\mathcal{L}} \exp(-cs^\nu), & \Re(s) > 0. & \text{(c)} \end{cases} \quad (4.26)$$

We note that, thanks to Bernstein theorem, Eq. (4.26-a) proves the positivity of  $M_\nu(r)$  for  $r > 0$  since for  $0 < \nu < 1$  the function  $E_\nu$  is known to be completely monotonic on the negative real axis. Furthermore, in the

(singular) limit for  $\nu \rightarrow 1^-$ , Eq. (4.26-a) shows that  $M_\nu(r)$  tends to the Dirac function  $\delta(r-1)$ . Mainardi, see *e.g.* [27, 28], has provided plots for  $r > 0$ , which show that  $M_\nu(r)$  is indeed positive, and is monotonically decreasing for  $0 < \nu \leq 1/2$ , while for  $1/2 < \nu < 1$  it exhibits a maximum whose position tends to  $r = 1$  as  $\nu \rightarrow 1^-$ , consistently with the limiting case  $M_1(r) = \delta(r-1)$ . We recognize that the result in (4.23) obtained as Laplace inversion of (4.21) is included in the Laplace transform pair (4.26-b) by taking  $\nu = \beta/2$ ,  $c = |x|$  and  $r = t$ .

For the function  $M_\nu(r)$  it is worth to recall the asymptotic representation

$$M_\nu(r) \sim A_0 Y^{\nu-1/2} \exp(-Y), \quad r \rightarrow \infty, \quad (4.27)$$

$$A_0 = \frac{1}{\sqrt{2\pi} (1-\nu)^\nu \nu^{2\nu-1}}, \quad Y = (1-\nu) (\nu^\nu r)^{1/(1-\nu)},$$

and the integral

$$\int_0^{+\infty} r^\delta M_\nu(r) dr = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}, \quad \delta > -1. \quad (4.28)$$

As a consequence of Eqs. (4.23) and (4.27)  $G_{2,\beta}^0(x,t)$  with  $0 < \beta < 2$  can be interpreted as a symmetric spatial *pdf* evolving in time, with a stretched exponential decay. More precisely, we have

$$G_{2,\beta}^0(x,1) = \frac{1}{2} M_{\beta/2}(|x|) \sim A x^a e^{-bx^c}, \quad x \rightarrow +\infty, \quad (4.29)$$

$$A = \left\{ 2\pi(2-\beta) 2^{\beta/(2-\beta)} \beta^{(2-2\beta)/(2-\beta)} \right\}^{-1/2}, \quad (4.30)$$

$$a = \frac{2\beta-2}{2(2-\beta)}, \quad b = (2-\beta) 2^{-2/(2-\beta)} \beta^{\beta/(2-\beta)}, \quad c = \frac{2}{2-\beta}.$$

Furthermore, using (4.28), the moments (of even order) of  $G_{2,\beta}^0(x,t)$  are

$$\mu_{2n}(t) := \int_{-\infty}^{+\infty} x^{2n} G_{2,\beta}^0(x,t) dx = \frac{\Gamma(2n+1)}{\Gamma(\beta n+1)} t^{\beta n}, \quad n = 0, 1, 2, \dots, t \geq 0. \quad (4.31)$$

It is interesting to compare this expression with the analogue (4.2) for the Gaussian. In particular, the variance  $\sigma^2 := \mu_2 = 2t^\beta/\Gamma(\beta+1)$  is now proportional to the  $\beta$ -th power of time, consistent with *anomalous slow diffusion* for  $0 < \beta < 1$  and *anomalous fast diffusion* for  $1 < \beta < 2$ .

In the slow diffusion case ( $0 < \beta < 1$ ) the *pdf* attains its maximum value at  $x = 0$  (where the first derivative is discontinuous) and exhibits exponential tails fatter than the Gaussian; in the fast diffusion case ( $1 < \beta < 2$ ) the *pdf* attains two symmetric maxima that move apart from the origin with time and exhibits exponential tails thinner than the Gaussian.

We also outline some other interesting properties of the  $M$  function, which put it in relationship with the class of extremal stable distributions. In fact, we note that the Laplace transform pair (4.26-c), compared with (4.5), allows us to relate the one-sided, extremal, Lévy stable *pdf* of index  $\nu$  ( $0 < \nu < 1$ ) with the  $M$  function of order  $\nu$ ; we have

$$\frac{1}{c^{1/\nu}} L_\nu^{-\nu} \left( \frac{r}{c^{1/\nu}} \right) = \frac{c\nu}{r^{\nu+1}} M_\nu \left( \frac{c}{r^\nu} \right), \quad 0 < \nu < 1, \quad c > 0, \quad r > 0. \quad (4.32)$$

Incidentally, the above relation turns out to provide an alternative proof of the non-negativity of the  $M$  function for positive argument. Furthermore, putting in Eqs (4.28) and (4.32)  $\nu = \alpha$ ,  $c = t > 0$  and  $r = x > 0$ , we get the Laplace transform pair related to the (spatial one-sided) extremal stable *pdf* (evolving in time) of index  $\alpha$  ( $0 < \alpha < 1$ ), which represents the Green function of the space-fractional diffusion equation with  $\theta = -\alpha$ , see Eqs. (4.4)-(4.5), and henceforth,

$$e^{-ts^\alpha} \stackrel{\mathcal{L}}{\leftrightarrow} t^{-1/\alpha} L_\alpha^{-\alpha}(x/t^{1/\alpha}) = G_{\alpha,1}^{-\alpha}(x,t) = \alpha \frac{t}{x^{\alpha+1}} M_\alpha(t/x^\alpha), \quad (4.33)$$

with  $0 < \alpha < 1$ ,  $x > 0$ ,  $t > 0$ .

Then, in virtue of the Feller reciprocity relation (4.19) and using Eqs. (4.4) and (4.23), after some manipulations we can recover the noteworthy result

$$G_{2,\beta}^0(x,t) = \frac{1}{\beta} G_{2/\beta,1}^{2/\beta-2}(x,t), \quad 1 < \beta < 2, \quad x > 0, \quad t > 0. \quad (4.34)$$

This result (already noted by Mainardi & Tomirotti [31] by comparing the corresponding series expansions, see (4.13) and (4.24)) states that the two symmetric branches of the Green function of the time-fractional diffusion equation of order  $1 < \beta < 2$  are proportional to the corresponding (not symmetric) branches of the Green function of the space-fractional diffusion equation of order  $1 < \alpha = 2/\beta < 2$  with skewness  $\theta = \pm(2 - \beta/2)$ , namely the exponential queues of the two extremal stable distributions of index  $2/\beta$ .

In the limit  $\beta = 2$  we recover the fundamental solution of the D'Alembert wave equation, *i.e.*

$$G_{2,2}^0(x,t) = \frac{\delta(x-t) + \delta(x+t)}{2} = \frac{G_{1,1}^{-1}(x,t) + G_{1,1}^{+1}(x,t)}{2}, \quad (4.35)$$

where  $-\infty < x < +\infty$ ,  $t \geq 0$ .

At the end of the paper we shall exhibit some plots the fundamental solution of the time-fractional diffusion equation,

$$G_{2,\beta}^0(x,1) = K_{2,\beta}^0(x) = \frac{1}{2} M_{\beta/2}(x), \quad 0 < \beta < 2, \quad (4.36)$$

in the range  $|x| \leq 5$ .

### The neutral-fractional diffusion

Let us now consider  $\{0 < \alpha = \beta \leq 2\}$  (*neutral fractional diffusion*), which includes the Cauchy diffusion for  $\alpha = \beta = 1$  ( $\theta = 0$ ) and the limiting case of wave propagation for  $\alpha = \beta = 2$ .

For  $1 \leq \alpha < 2$  and  $\theta = 0$  the Green function has been derived in explicit form by Gorenflo, Iskenderov and Luchko [17] by using the Mellin-Barnes integral representation. We shall adopt this representation later. Now, we consider it conceptually more economical to remain (as long as possible) in the kingdom of Fourier-Laplace transforms and we point out the following Fourier transform pair related to the Mittag-Leffler function of our interest:

$$E_\alpha(-|\kappa|^\alpha) \stackrel{\mathcal{F}}{\leftrightarrow} \frac{1}{\pi} \frac{|x|^{\alpha-1} \sin(\alpha\pi/2)}{1 + 2|x|^\alpha \cos(\alpha\pi/2) + |x|^{2\alpha}}, \quad 0 < \alpha < 2, \quad x \in \mathbb{R}. \quad (4.37)$$

This pair can be verified as an exercise in complex analysis following the method illustrated by Gorenflo and Mainardi, see *e.g.* [21]<sup>7</sup>. As far as we know, this case of fractional diffusion seems not be treated in its greatest generality in the literature. Now, taking into account elementary properties

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<sup>7</sup> In [21] the reader can find the proof of the Laplace transform pair

$$E_\nu(-s^\nu) \stackrel{\mathcal{L}}{\leftrightarrow} \frac{1}{\pi} \frac{t^{\nu-1} \sin(\nu\pi)}{1 + 2t^\nu \cos(\nu\pi) + t^{2\nu}}, \quad 0 < \nu < 1, \quad t > 0, \quad \Re(s) > 0.$$

of the Fourier transform and (4.37), the reduced Green function in (3.19) reads for  $\alpha = \beta$  and  $x > 0$  as

$$K_{\alpha,\alpha}^{\theta}(x) = N_{\alpha}^{\theta}(x) = \frac{1}{\pi} \frac{x^{\alpha-1} \sin[\frac{\pi}{2}(\alpha - \theta)]}{1 + 2x^{\alpha} \cos[\frac{\pi}{2}(\alpha - \theta)] + x^{2\alpha}}, \quad 0 < \alpha < 2. \quad (4.38)$$

This solution, that can be extended to negative  $x$  by setting  $N_{\alpha}^{\theta}(-x) = N_{\alpha}^{-\theta}(x)$ , is evidently not negative in all of  $\mathbb{R}$ , so it can be interpreted as a *probability density*. In other words,  $N_{\alpha}^{\theta}(x)$  may be considered the fractional generalization with skewness of the well-known Cauchy density (4.8). In the limiting case  $\alpha \rightarrow 2^-$  (with  $\theta = 0$ ) the density tends to the combination  $[\delta(x-1) + \delta(x+1)]/2$ , so we recover the Green function of the D'Alembert wave equation quoted in (4.35).

## 5. Composition rule for the Green function with $0 < \beta \leq 1$

We now present a composition rule which allows us to express the general Green function of the space-time fractional diffusion equation (with the restriction  $0 < \beta \leq 1$ ) as an integral involving the two Green functions corresponding to space-fractional and time-fractional diffusion equations. To this purpose we note that the Fourier Laplace transform of the Green function (3.5) can be re-written in integral form as in [38]

$$\begin{aligned} \widehat{G}_{\alpha,\beta}^{\theta}(\kappa, s) &= \frac{s^{\beta-1}}{s^{\beta} + \psi_{\alpha}^{\theta}(\kappa)} = s^{\beta-1} \int_0^{\infty} e^{-u[s^{\beta} + \psi_{\alpha}^{\theta}(\kappa)]} du \\ &= \int_0^{\infty} e^{-u\psi_{\alpha}^{\theta}(\kappa)} \left( s^{\beta-1} e^{-us^{\beta}} \right) du. \end{aligned} \quad (5.1)$$

In view of Eqs. (4.3) and (4.26) we can interpret the above formula as

$$\widehat{G}_{\alpha,\beta}^{\theta}(\kappa, s) = 2 \int_0^{\infty} \widehat{G}_{\alpha,1}^{\theta}(\kappa, u) \widetilde{G}_{2,2\beta}^0(u, s) du. \quad (5.2)$$

Then, by inversion, we obtain the required composition rule

$$G_{\alpha,\beta}^{\theta}(x, t) = 2 \int_0^{\infty} G_{\alpha,1}^{\theta}(x, u) G_{2,2\beta}^0(u, t) du. \quad (5.3)$$



Note the presence of  $G_{2,2\beta}^0$  instead of  $G_{2,\beta}^0$ . Hence Eq. (5.3) is a formula separating variables. It states that the Green function for the space-time-fractional diffusion equation of order  $\{\alpha, \beta\}$ , with  $0 < \alpha \leq 2$  and  $0 < \beta \leq 1$ , can be expressed in terms of the Green function for the space-fractional diffusion equation of order  $\alpha$  and the Green function for the time-fractional diffusion equation of order  $2\beta$ .

We now present alternative, equivalent forms of the composition rule, that directly involve the functions  $L$  and  $M$ , limiting ourselves to  $x > 0$ . Because of Eqs. (4.4) and (4.23), we can write Eq. (5.3) as

$$G_{\alpha,\beta}^\theta(x,t) = t^{-\beta} \int_0^\infty u^{-1/\alpha} L_\alpha^\theta\left(\frac{x}{u^{1/\alpha}}\right) M_\beta\left(\frac{u}{t^\beta}\right) du. \quad (5.4)$$

REMARK 5.1. The formulae (4.4) and (4.23) corresponding to the particular cases  $\{\alpha, 1\}$  (*space-fractional diffusion*) and  $\{2, \beta\}$  (*time-fractional diffusion*) are recovered from (5.4) as follows:

$$G_{\alpha,1}^\theta(x,t) = t^{-1} \int_0^\infty u^{-1/\alpha} L_\alpha^\theta\left(\frac{x}{u^{1/\alpha}}\right) \delta\left(\frac{u}{t} - 1\right) du = t^{-1/\alpha} L_\alpha^\theta\left(\frac{x}{t^{1/\alpha}}\right), \quad (5.5)$$

$$G_{2,\beta}^0(x,t) = \frac{t^{-\beta}}{2\sqrt{\pi}} \int_0^\infty e^{-x^2/(4u)} M_\beta\left(\frac{u}{t^\beta}\right) \frac{du}{u^{1/2}} = \frac{t^{-\beta/2}}{2} M_{\beta/2}\left(\frac{|x|}{t^{\beta/2}}\right). \quad (5.6)$$

Eq. (5.6) is of high interest for the theory of the  $M$  functions since it is a sort of (integral) duplication formula with respect to the order; it is worth to note the presence of the Gaussian with a spreading variance in the kernel of the integral.

Taking into account the relation (4.32), the function  $M$  in (5.4) can be expressed in terms of an  $L$  function. We have

$$G_{\alpha,\beta}^\theta(x,t) = \int_0^\infty \left[ u^{-1/\alpha} L_\alpha^\theta\left(\frac{x}{u^{1/\alpha}}\right) \right] \left[ \frac{t}{u\beta} u^{-1/\beta} L_\beta^{-\beta}\left(\frac{t}{u^{1/\beta}}\right) \right] du. \quad (5.7)$$

Putting  $y = u^{-1/\beta}$  we derive from (5.7) the composition rule in the form recently obtained by Uchaikin, see [46, 47], (for  $t = 1$ ):

$$G_{\alpha,\beta}^\theta(x,1) = K_{\alpha,\beta}^\theta(x) = \int_0^\infty y^{\beta/\alpha} L_\alpha^\theta\left(x y^{\beta/\alpha}\right) L_\beta^{-\beta}(y) dy. \quad (5.8)$$

The composition rule, in its equivalent forms (5.3), (5.4), (5.8), shows that the Green function of the space-time fractional diffusion equation is non

negative for any  $x \in \mathbb{R}$  and  $t \in \mathbb{R}_0^+$ , when  $\{0 < \alpha < 2, 0 < \beta \leq 1\}$ . We also know that this is the case when  $\{\alpha = 2, 0 < \beta \leq 2\}$  (*time-fractional diffusion*) and when  $\{0 < \alpha = \beta \leq 2\}$  (*neutral-fractional diffusion*). In the next Section, by using the Mellin convolution, we shall derive a new composition rule which ensures the non-negativity of  $K_{\alpha,\beta}^\theta(x)$  in all of  $\mathbb{R}$  for  $\{0 < \alpha \leq 2\} \cap \{0 < \beta/\alpha \leq 1\}$ . Then we can conclude that the Green function can be surely interpreted as a spatial *pdf*, evolving in time, in the ranges  $\{0 < \alpha \leq 2\} \cap \{0 < \beta \leq 1\}$  and  $\{0 < \beta \leq \alpha \leq 2\}$ .

## 6. Mellin-Barnes integral representations for the Green function

Let us now consider the Fourier representation of the general Green function  $K_{\alpha,\beta}^\theta(x)$  (restricting to  $x > 0$ ) as stated at the end of the Sect 3, see Eqs (3.18)-(3.21). We intend to use the Mellin convolution to invert the relevant Fourier transforms according to the scheme (2.26)-(2.31). To this purpose we also need the Mellin transform pair deduced from (3.12)-(3.13)

$$E_\beta\left(-\kappa^\alpha e^{i\theta\pi/2}\right) \stackrel{\mathcal{M}}{\leftrightarrow} \frac{1}{\alpha} \frac{\Gamma(\frac{s}{\alpha}) \Gamma(1 - \frac{s}{\alpha})}{\Gamma(1 - \frac{\beta}{\alpha}s)} \exp\left(-i\frac{\theta\pi}{2} \frac{s}{\alpha}\right), \quad (6.1)$$

where  $\kappa > 0$ ,  $|\theta| \leq 2 - \beta$ ,  $0 < \Re(s) < \alpha$ .

Using Eq. (6.1) in Eqs (2.28) and (2.29), Eq. (3.12) yields

$$K_{\alpha,\beta}^\theta(x) = {}_cK_{\alpha,\beta}^\theta(x) + {}_sK_{\alpha,\beta}^\theta(x), \quad x > 0, \quad (6.2)$$

with

$$\begin{aligned} {}_cK_{\alpha,\beta}^\theta(x) &= \frac{1}{\pi\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha}) \Gamma(1 - \frac{s}{\alpha}) \Gamma(1-s)}{\Gamma(1 - \frac{\beta}{\alpha}s)} \sin\left(\frac{\pi s}{2}\right) \cos\left(\frac{\theta\pi s}{2\alpha}\right) x^s ds, \\ {}_sK_{\alpha,\beta}^\theta(x) &= -\frac{1}{\pi\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha}) \Gamma(1 - \frac{s}{\alpha}) \Gamma(1-s)}{\Gamma(1 - \frac{\beta}{\alpha}s)} \cos\left(\frac{\pi s}{2}\right) \sin\left(\frac{\theta\pi s}{2\alpha}\right) x^s ds, \end{aligned} \quad (6.3)$$

where  $0 < \gamma < \min\{\alpha, 1\}$ , and  $|\theta| \leq 2 - \beta$ . We thus obtain from Eqs (6.2)-(6.3)

$$K_{\alpha,\beta}^\theta(x) = \frac{1}{\pi\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha}) \Gamma(1 - \frac{s}{\alpha}) \Gamma(1-s)}{\Gamma(1 - \frac{\beta}{\alpha}s)} \sin\left[\frac{s}{\alpha} \frac{\pi}{2} (\alpha - \theta)\right] x^s ds. \quad (6.4)$$

By setting

$$\rho = \frac{\alpha - \theta}{2\alpha}, \quad (6.5)$$

and using the reflection formula for the gamma function, we finally obtain

$$K_{\alpha,\beta}^{\theta}(x) = \frac{1}{\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha}) \Gamma(1 - \frac{s}{\alpha}) \Gamma(1-s)}{\Gamma(1 - \frac{\beta}{\alpha}s) \Gamma(\rho s) \Gamma(1-\rho s)} x^s ds. \quad (6.6)$$

By using a standard notation for the "gamma fraction", see *e.g.* [32], p. 65 (4.13), we can re-write (6.6) as

$$K_{\alpha,\beta}^{\theta}(x) = \frac{1}{\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma \left[ \frac{s}{\alpha}, 1 - \frac{s}{\alpha}, 1-s \right] x^s ds. \quad (6.6')$$

The formulae (6.4) and (6.6) are valid for  $0 < \gamma < \min\{\alpha, 1\}$  with  $|\theta| \leq 2 - \beta$ , and provide equivalent integral representations of the Green function for the general *space-time fractional diffusion* equation. In the following we shall use (6.4) or (6.6) [(6.6')] according to our convenience.

The integral at the RHS of Eq. (6.6) is a particular *Mellin-Barnes integral*<sup>8</sup>, according to a usual terminology, see *e.g.* [11], Vol. 1, Ch. 1, §1.19, pp. 49-50. The interested reader can find in [11] the discussion on the general conditions of convergence for the typical Mellin-Barnes integral, formerly given by Dixon & Ferrar [9], and based on the asymptotic representation of the gamma function. By using these results we can verify once again the convergence condition (3.17), namely  $|\theta| \leq 2 - \beta$ .

In the particular cases of *standard diffusion*, *space-fractional diffusion*, *time-fractional diffusion* and *neutral-fractional diffusion*, the Mellin-Barnes integrals in Eq. (6.6) simplify as follows.

For the *standard diffusion*, namely for  $\{\alpha = 2, \theta = 0, \beta = 1\}$ , we have  $\rho = 1/2$  and

$$K_{2,1}^0(x) = L_2^0(x) = \frac{1}{2x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-s)}{\Gamma(1-s/2)} x^s ds, \quad 0 < \gamma < 1. \quad (6.7)$$

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<sup>8</sup>The names refer to the two authors, who in the first 1910's developed the theory of these integrals using them for a complete integration of the hypergeometric differential equation. However, as pointed out by Tricomi in [11] (Vol. 1, Ch. 1, §1.19, p. 49), these integrals were first used by S. Pincherle in 1888. For a revisited analysis of the pioneering work of Pincherle (1853-1936, Professor of Mathematics at the University of Bologna from 1880 to 1928) we refer to the recent paper by Mainardi and Pagnini [30].

For the space-fractional diffusion ( $\{0 < \alpha < 2, |\theta| \leq \min\{\alpha, 2 - \alpha\}, \beta = 1\}$ ) we distinguish a number of cases.

For  $\{0 < \alpha < 1, |\theta| < \alpha, \beta = 1\}$  we have  $0 < \rho < 1$  and

$$K_{\alpha,1}^{\theta}(x) = L_{\alpha}^{\theta}(x) = \frac{1}{\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s/\alpha) \Gamma(1-s)}{\Gamma(\rho s) \Gamma(1-\rho s)} x^s ds, \quad 0 < \gamma < \alpha. \quad (6.8)$$

Let us consider the extremal cases. For  $\theta = -\alpha$ , we have  $\rho = 1$  so  $\Gamma(1-s)$  cancels with  $\Gamma(1-\rho s)$ . Then the "gamma fraction" in (6.8) reduces to  $\Gamma(s/\alpha)/\Gamma(s)$ . For  $\theta = \alpha$ , we have  $\rho = 0$  so  $K_{\alpha,1}^{\theta}(x) = 0$  for  $x > 0$ .

For  $\{\alpha = 1, |\theta| < 1, \beta = 1\}$  we have  $0 < \rho < 1$  and

$$K_{1,1}^{\theta}(x) = L_1^{\theta}(x) = \frac{1}{x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s) \Gamma(1-s)}{\Gamma(\rho s) \Gamma(1-\rho s)} x^s ds, \quad 0 < \gamma < 1. \quad (6.9)$$

Let us consider the extremal cases. For  $\theta = -1$ , we have  $\rho = 1$  so  $\Gamma(1-s)$  cancels with  $\Gamma(1-\rho s)$  and  $\Gamma(s)$  cancels with  $\Gamma(\rho s)$ . Then the "gamma fraction" in (6.9) reduces to 1 to yield the Mellin representation of the Dirac function  $\delta(x-1)$ . For  $\theta = 1$ , we have  $\rho = 0$  so  $K_{1,1}^{\theta}(x) = 0$  for  $x > 0$ .

For  $\{1 < \alpha < 2, |\theta| < 2 - \alpha, \beta = 1\}$  we have  $0 < (\alpha-1)/\alpha < \rho < 1/\alpha < 1$  and

$$K_{\alpha,1}^{\theta}(x) = L_{\alpha}^{\theta}(x) = \frac{1}{\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s/\alpha) \Gamma(1-s)}{\Gamma(\rho s) \Gamma(1-\rho s)} x^s ds, \quad 0 < \gamma < 1. \quad (6.10)$$

Let us consider the extremal cases. For  $\theta = -(2-\alpha)$ , we have  $\rho = (\alpha-1)/\alpha > 0$  so (6.10) is still valid. For  $\theta = 2-\alpha$  we have  $\rho = 1/\alpha$  so  $\Gamma(s/\alpha)$  cancels with  $\Gamma(\rho s)$ , and the "gamma fraction" in (6.10) reduces to  $\Gamma(1-s)/\Gamma(1-s/\alpha)$ .

For the time-fractional diffusion ( $\{\alpha = 2, \theta = 0, 0 < \beta < 2, \beta \neq 1\}$ ) we put  $\alpha = 2$  and  $\rho = 1/2$  in (6.6) and obtain

$$K_{2,\beta}^0(x) = \frac{1}{2} M_{\beta/2}(x) = \frac{1}{2x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-s)}{\Gamma(1-\beta s/2)} x^s ds, \quad 0 < \gamma < 1. \quad (6.11)$$

For the neutral-fractional diffusion ( $\{0 < \alpha = \beta < 2\}$ ) we put  $\alpha = \beta$  in (6.6) and obtain

$$K_{\alpha,\alpha}^{\theta}(x) = N_{\alpha}^{\theta}(x) = \frac{1}{\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha}) \Gamma(1-\frac{s}{\alpha})}{\Gamma(\rho s) \Gamma(1-\rho s)} x^s ds. \quad (6.12)$$

In the limiting case  $\alpha = \beta = 2$  ( $\theta = 0$ ,  $\rho = 1/2$ ) we recover from (6.11) or (6.12) the Mellin-Barnes representation of  $\delta(x-1)/2$ , consistently with the D'Alembert wave equation, see Eq. (4.35). We note that the Mellin transform pair (6.6) allows us to compute the value of any convergent moment of the Green function  $K_{\alpha,\beta}^\theta(x)$ . In fact, recalling the basic formulae (2.21a)-(2.21b) for the Mellin transformation (2.21a)-(2.21b), from (6.6) we write the Mellin transform of  $x K_{\alpha,\beta}^\theta(x)$  as

$$\int_0^{+\infty} K_{\alpha,\beta}^\theta(x) x^s dx = \frac{1}{\alpha} \frac{\Gamma(-s/\alpha) \Gamma(1+s/\alpha) \Gamma(1+s)}{\Gamma(1+\beta s/\alpha) \Gamma(-\rho s) \Gamma(1+\rho s)}, \quad (6.13)$$

$$-\min\{\alpha, 1\} < \Re(s) < 0.$$

In order to include  $s = 0$  in the convergence strip (so, in particular, the integral of  $K_{\alpha,\beta}^\theta(x)$  in  $\mathbb{R}_0^+$  can be evaluated) we properly use in (6.13) the known property  $\Gamma(1+z) = z\Gamma(z)$  to obtain

$$\int_0^{+\infty} K_{\alpha,\beta}^\theta(x) x^s dx = \rho \frac{\Gamma(1-s/\alpha) \Gamma(1+s/\alpha) \Gamma(1+s)}{\Gamma(1-\rho s) \Gamma(1+\rho s) \Gamma(1+\beta s/\alpha)}, \quad (6.14)$$

$$-\min\{\alpha, 1\} < \Re(s) < \alpha.$$

In particular, setting  $s = 0$  we find  $\int_0^{+\infty} K_{\alpha,\beta}^\theta(x) dx = \rho$  ( $\rho = 1/2$  if  $\theta = 0$ ). Eq. (6.14) is consistent with a similar expression given by Uchaikin [47]. We note that it is strictly valid as soon as cancellations in the "gamma fraction" at the RHS cannot be done. Then Eq. (6.14) allows us to evaluate (in  $\mathbb{R}_0^+$ ) the (absolute) moments of the fundamental solution of order  $\delta$  such that  $-\min\{\alpha, 1\} < \delta < \alpha$ . In other words, it states that, as  $x \rightarrow +\infty$ ,  $K_{\alpha,\beta}^\theta(x) = \mathcal{O}(x^{-(\alpha+1)})$ . When cancellations occur in the "gamma fraction" the range of  $\delta$  may change. In this respect an interesting case is  $\{\alpha = 2, \theta = 0, 0 < \beta < 2\}$  (*time-fractional diffusion including standard diffusion*), where Eq. (6.14) reduces to

$$\int_0^{+\infty} K_{2,\beta}^0(x) x^s dx = \frac{1}{2} \frac{\Gamma(1+s)}{\Gamma(1+\beta s/2)}, \quad \Re(s) > -1. \quad (6.15)$$

This result is consistent with the existence of all moments of order  $\delta > -1$  for the corresponding Green function, see (4.28).

Furthermore, by using the Mellin convolution formula (2.25), and the Mellin-Barnes representations valid in the particular cases, namely (6.8), (6.10) for the space-fraction diffusion, (6.11) for the time-fraction diffusion and (6.12) for the neutral-fraction diffusion, Eq. (6.6) can help us to investigate the non-negativity of the Green function  $K_{\alpha,\beta}^\theta(x)$ . In fact, it is not difficult to recognize that, assuming  $x > 0$  and  $0 < \alpha < 2$ ,

$$K_{\alpha,\beta}^\theta(x) = \begin{cases} \alpha \int_0^\infty [\xi^{\alpha-1} M_\beta(\xi^\alpha)] L_\alpha^\theta(x/\xi) \frac{d\xi}{\xi}, & 0 < \beta < 1, \\ \int_0^\infty M_{\beta/\alpha}(\xi) N_\alpha^\theta(x/\xi) \frac{d\xi}{\xi}, & 0 < \beta/\alpha < 1. \end{cases} \quad (6.16)$$

The limiting cases  $\beta = 1$ ,  $\alpha = \beta$  and  $\alpha = 2$  can be included if one takes into account, in a proper way, the resulting expressions by the generalized function  $\delta(x - 1)$ . We note that the first Mellin convolution in (6.16) is consistent with the composition rules obtained in Sect 5 by using the Fourier transform<sup>9</sup>.

The formulae in (6.16), by involving two non negative functions (related to the probability densities  $L_\alpha^\theta(x)$ ,  $M_{\beta/2}(x)$  or  $N_\alpha^\theta(x)$ ), allow us to state the probability interpretation of the Green function in the ranges

$$\{0 < \alpha \leq 2\} \cap \{0 < \beta \leq 1\}$$

and

$$\{0 < \alpha \leq 2\} \cap \{0 < \beta/\alpha \leq 1\}.$$

Therefore, the probability interpretation holds true for any  $\alpha \in (0, 2]$  if  $0 < \beta \leq 1$ , whereas, if  $1 < \beta \leq 2$ , only for  $1 < \beta \leq \alpha \leq 2$ . The cases excluded from the probability interpretation turn out to be

$$\{0 < \alpha < \beta\} \cap \{1 < \beta < 2\}.$$

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<sup>9</sup>To this purpose, let us consider the composition rule in form (5.4),

$$G_{\alpha,\beta}^\theta(x, t) = t^{-\beta} \int_0^\infty u^{-1/\alpha} L_\alpha^\theta\left(\frac{x}{u^{1/\alpha}}\right) M_\beta\left(\frac{u}{t^\beta}\right) du,$$

Putting  $\xi = u^{1/\alpha}$  and taking into account the scaling property  $G_{\alpha,\beta}^\theta(x, t) = t^{-\beta/\alpha} K_{\alpha,\beta}^\theta(x/t^{\beta/\alpha})$ , we rewrite the composition rule (for  $t = 1$ ) in the form of the first Eq. in (6.16).

## 7. Computational representations for the Green function

Those who are acquainted with Fox  $H$  functions can recognize in (6.6) the representation of a certain function of this class<sup>10</sup> on which the interested reader can inform himself in several books, *e.g.* [33], [44], [32], [37], [39], [26]. Unfortunately, as far as we know, computing routines for this general class of special functions are not yet available. Here, following and completing the approach adopted by Gorenflo, Iskenderov & Luchko [17], we intend to compute the Green function in any space domain providing for it a *computational representation* (usually to be obtained by matching two distinct expressions).

To this purpose we distinguish three cases depending on the order relationship between  $\alpha$  and  $\beta$ . According to their increasing difficulty we consider

$$(i) \quad \alpha = \beta, \quad (ii) \quad \alpha < \beta, \quad (iii) \quad \alpha > \beta. \quad (7.1)$$

For the case (i)  $\{\alpha = \beta\}$  the integral representation (6.4) simplifies into

$$K_{\alpha,\alpha}^{\theta}(x) = \frac{1}{\pi\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(\frac{s}{\alpha}\right) \Gamma\left(1 - \frac{s}{\alpha}\right) \sin\left[\frac{s}{\alpha} \frac{\pi}{2} (\alpha - \theta)\right] x^s ds. \quad (7.2)$$

In this case the contour of integration can be transformed to the loop  $L_{-\infty}$  starting and ending at infinity and encircling all the poles  $s_n = -\alpha n$ ,  $n = 0, 1, 2, \dots$  of the function  $\Gamma(s/\alpha)$  for  $0 < x < 1$  and to the loop  $L_{+\infty}$  starting and ending at infinity and encircling all the poles  $s_n = (1+n)\alpha$ ,  $n = 0, 1, 2, \dots$  of the function  $\Gamma(1 - s/\alpha)$  for  $x > 1$ . Applying the residue theorem we arrive at the series representations

$$K_{\alpha,\alpha}^{\theta}(x) = \frac{1}{\pi x} \sum_{n=0}^{\infty} \sin\left[\frac{n\pi}{2}(\theta - \alpha)\right] (-x^{\alpha})^n, \quad 0 < x < 1; \quad (7.3)$$

$$K_{\alpha,\alpha}^{\theta}(x) = \frac{1}{\pi x} \sum_{n=0}^{\infty} \sin\left[\frac{n\pi}{2}(\theta - \alpha)\right] (-x^{-\alpha})^n, \quad 1 < x < \infty. \quad (7.4)$$

So we have two different representations by power series: the one in *positive powers*, convergent for  $0 < x < 1$ , the other in *negative powers*, convergent

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<sup>10</sup>ADDED NOTE (2007) As a complement of the present analysis the reader can consult the paper by F. Mainardi, G. Pagnini and R.K. Saxena: Fox  $H$  functions in fractional diffusion, published in *Journal of Computational and Applied Mathematics* **178** (2005) 321-331, where the Authors have provided the representation of the fundamental solutions of the fractional diffusion equations here treated in terms of Fox  $H$  functions

for  $x > 1$ . Remarkably, in this special case, we can obtain the fundamental solution in all of  $\mathbb{R}$  in a closed form, namely expressed in term of elementary functions. In fact, following [17], we use the formula

$$\sum_{n=1}^{\infty} r^n \sin(na) = \Im \left( \sum_{n=1}^{\infty} r^n e^{ina} \right) = \Im \left( \frac{re^{ia}}{1 - re^{ia}} \right) = \frac{r \sin a}{1 - 2r \cos a + r^2}, \quad (7.5)$$

with  $a \in \mathbb{R}$ ,  $|r| < 1$ , and we get

$$K_{\alpha, \alpha}^{\theta}(x) = \frac{1}{\pi} \frac{x^{\alpha-1} \sin\left[\frac{\pi}{2}(\alpha - \theta)\right]}{1 + 2x^{\alpha} \cos\left[\frac{\pi}{2}(\alpha - \theta)\right] + x^{2\alpha}}, \quad 0 < x < \infty. \quad (7.6)$$

This result is in agreement with (4.38) and, by taking into account the symmetry relation (3.18), can be properly extended to  $-\infty < x < 0$ . We also note that from (7.6) we recover the expressions of the particular cases  $\{\alpha = \beta = 1, |\theta| < 1\}$  given in Eq. (4.9), namely the stable density  $L_1^{\theta}(x)$ , and  $\{1 \leq \alpha = \beta < 2, \theta = 0\}$  given by Gorenflo, Iskenderov and Luchko, see Eq. (20) in [17]. In the limit as  $x \rightarrow 0$ , we get

$$\lim_{x \rightarrow 0} K_{\alpha, \alpha}^{\theta}(x) = \lim_{x \rightarrow 0} \frac{x^{\alpha-1}}{\pi} \sin \left[ \frac{\pi}{2}(\alpha - \theta) \right] = \begin{cases} +\infty, & 0 < \alpha < 1, \\ \frac{1}{\pi} \cos\left[\frac{\pi}{2}\theta\right], & \alpha = 1, \\ 0, & 1 < \alpha < 2, \end{cases} \quad (7.7)$$

which is consistent with (3.23).

For the case (ii)  $\{\alpha < \beta\}$  the contour of integration in (6.6) can be transformed to the loop  $L_{-\infty}$  starting and ending at infinity and encircling all the poles  $s_n = -\alpha n$ ,  $n \in \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$  of the function  $\Gamma(s/\alpha)$ . The residue theorem for simple poles gives us the following representation by a convergent series in *negative powers* of  $x^{\alpha}$ ,

$$K_{\alpha, \beta}^{\theta}(x) = \frac{1}{\pi x} \sum_{n=1}^{\infty} \frac{\Gamma(1 + \alpha n)}{\Gamma(1 + \beta n)} \sin \left[ \frac{n\pi}{2}(\theta - \alpha) \right] (-x^{-\alpha})^n, \quad 0 < x < \infty. \quad (7.8)$$

We note that for  $\beta = 1$  we recover the series at the RHS of (4.12) giving the representation of the stable density  $L_{\alpha}^{\theta}(x)$  with  $0 < \alpha < 1$ .

For the case (iii)  $\{\alpha > \beta\}$  the situation is more complicated because we have to transform the contour of integration in (6.6) to the loop  $L_{+\infty}$  encircling all the poles  ${}_1s_n = \alpha(1 + n)$ ,  $n \in \mathbb{N}_0$  and  ${}_2s_m = 1 + m$ ,  $m \in \mathbb{N}_0$  of the functions  $\Gamma(1 - \frac{s}{\alpha})$  and  $\Gamma(1 - s)$ , respectively.



We note that for  $\beta = 1$  or  $\alpha = 2$  the poles  ${}_1s_n$  do not appear because  $\Gamma(1 - s/\alpha)$  cancels in (6.6) as seen in Eqs (6.7)-(6.11).

Excluding the above cases, *i.e.* if  $\beta \neq 1$  and  $\alpha \neq 2$ , we have to consider the possibility of double poles occurring when  $-\alpha(k+1) = -(m+1)$ , namely when  $\alpha = (m+1)/(k+1)$ ,  $m, k \in N_0$ .

When the poles are all simple the residue theorem gives us the following representation by two convergent series in *positive powers* of  $x^\alpha$  and  $x$ ,

$$K_{\alpha,\beta}^\theta(x) = \frac{1}{\pi x} \sum_{k=0}^{\infty} \frac{\Gamma(1 - \alpha k)}{\Gamma(1 - \beta k)} \sin \left[ \frac{k\pi}{2}(\theta - \alpha) \right] (-x^\alpha)^k$$

$$+ \frac{1}{\pi x} \sum_{k=0}^{\infty} \frac{\Gamma(1 - \frac{k}{\alpha})\Gamma(1 + \frac{k}{\alpha})}{k!\Gamma(1 - \frac{\beta}{\alpha}k)} \sin \left[ \frac{k\pi}{2\alpha}(\theta - \alpha) \right] (-x)^k. \quad (7.9)$$

We note that in the limiting cases  $\beta = 1$  or  $\alpha = 2$  the first series has vanishing coefficients and the second series yields the expected results, namely the expansion in positive powers of  $x$  for the stable density  $L_\alpha^\theta(x)$  with  $\alpha > 1$ , see (4.13), and for the density  $M_{\beta/2}(|x|)/2$ , see (4.24) and (4.36), respectively.

When the poles are not all simple, we need to separate the double poles from the simple ones. To this purpose we denote the *simple* poles of  $\Gamma(1 - s/\alpha)$  and  $\Gamma(1 - s)$  by  ${}_1s_k = \alpha(k+1)$ ,  $k \in K$ , and  ${}_2s_m = m+1$ ,  $m \in M$ , and the *double* poles of  $\Gamma(1 - \frac{s}{\alpha})\Gamma(1 - s)$  by  ${}_3s_i = (m_0+1)(1+i)$ ,  $i \in N_0$  where

$$K = N_0 \setminus K_0; N_0 = \{0, 1, 2, \dots\}, K_0 = \{k \in N \mid k = (k_0+1) + (k_0+1)i, i \in N_0\},$$

$$M = N_0 \setminus M_0; N_0 = \{0, 1, 2, \dots\}, M_0 = \{m \in N \mid m = (m_0+1) + (m_0+1)i, i \in N_0\}.$$

Then, applying the residues theorem for simple and double poles we arrive at the following representation with three series (of which the first in *positive powers* of  $x^\alpha$ , the second in *positive powers* of  $x$ , and the third of a peculiar character in that it contains *positive powers* of  $x$  that can be coupled with  $\log x$ ):

$$K_{\alpha,\beta}^\theta(x) = \frac{1}{\pi x} \sum_{k \in K} \frac{\Gamma(1 - \alpha k)}{\Gamma(1 - \beta k)} \sin \left[ \frac{k\pi}{2}(\theta - \alpha) \right] (-x^\alpha)^k$$

$$+ \frac{1}{\pi x} \sum_{m \in M} \frac{\Gamma(1 - \frac{m}{\alpha})\Gamma(1 + \frac{m}{\alpha})}{m!\Gamma(1 - \frac{\beta}{\alpha}m)} \sin \left[ \frac{m\pi}{2\alpha}(\theta - \alpha) \right] (-x)^m + P_{\alpha,\beta}^\theta(x). \quad (7.10)$$

Here  $P_{\alpha,\beta}^\theta(x)$  denotes the contribution from the double poles, which for  $\alpha = \frac{m_0+1}{k_0+1}$  reads (see detailed derivation in Pagnini's thesis [35]):

$$P_{\alpha,\beta}^\theta(x) = \frac{(-1)^{k_0+m_0+1} x^{m_0}}{\pi} \sum_{i=0}^{\infty} [(-1)^{k_0+m_0} x^{m_0+1}]^i \frac{R_1 + R_2 + R_3}{D} \times \sin \left[ \frac{(m_0+1)(1+i)\pi}{2\alpha} (\theta - \alpha) \right], \quad (7.11)$$

where

$$R_1 = \psi [(m_0+1)(1+i)] - \log x, \quad (7.12)$$

$$R_2 = -\frac{\beta}{\alpha} \psi [1 - \beta(k_0+1)(1+i)], \quad (7.13)$$

$$R_3 = -\frac{\pi}{2\alpha} (\theta - \alpha) \cot \left[ \frac{(m_0+1)(1+i)\pi}{2\alpha} (\theta - \alpha) \right], \quad (7.14)$$

$$D = \Gamma [1 - \beta(k_0+1)(1+i)] \Gamma [(m_0+1)(1+i)], \quad (7.15)$$

and  $\psi$  is the logarithmic derivative of the gamma function

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \quad (7.16)$$

**REMARK 6.1.** For some rational values of the parameter  $\beta$ , the relation  $1 - \beta(k_0+1)(2i+1) = -l$ ,  $l = 0, 1, 2, \dots$  can take place for some values of the index  $i$ ,  $i = 0, 1, 2, \dots$ . In this case, the term  $R_2/D$  in (7.11) is an indeterminate expression of the form  $\frac{\infty}{\infty}$ . Due to the formulae  $\lim_{s \rightarrow -l} 1/\Gamma(s) = 0$ ,  $l = 0, 1, 2, \dots$ , and  $\lim_{s \rightarrow -l} \psi(s)/\Gamma(s) = (-1)^{l+1} l!$ ,  $l = 0, 1, 2, \dots$ , we can rewrite  $P_{\alpha,\beta}^\theta(x)$  in the form (see detailed derivation in Pagnini's thesis [35]):

$$P_{\alpha,\beta}^\theta(x) = \frac{(-1)^{k_0+m_0+1} x^{m_0}}{\pi} \frac{\beta}{\alpha} \sum_{i=0}^{\infty} (-1)^{\beta(k_0+1)(1+i)+1} \frac{\Gamma [\beta(k_0+1)(1+i)]}{\Gamma [(m_0+1)(1+i)]} \times \sin \left[ \frac{(m_0+1)(1+i)\pi}{2\alpha} (\theta - \alpha) \right] [(-1)^{k_0+m_0} x^{m_0+1}]^i, \quad (7.17)$$

which gives us a finite value.

**REMARK 6.2.** From the above analysis we recognize that all series, that are involved in the cases (i), (ii), (iii), are vanishing if  $\theta = \alpha$ . Since this extremal value of  $\theta$  can be allowed only if  $0 < \alpha \leq 1$ , we can conclude that the Green

functions  $K_{\alpha,\beta}^\alpha(x) \equiv 0$  for  $x > 0$  if  $\alpha \in (0, 1]$  and  $\beta \in (0, 2)$ , subjected to the condition  $\theta = \alpha \leq 2 - \beta$ .

We must note that the representation of the Green function expressed for the cases (ii) and (iii) in terms of convergent expansions in negative and positive power series, respectively, are correspondingly not suitable to numerically evaluate the solution as soon as  $x$  is sufficiently small or sufficiently large. In other words, the convergent expansions cannot reproduce the expected behaviour of their sum for  $x \rightarrow 0^+$  or  $x \rightarrow \infty$  when the corresponding series are in negative or positive powers of  $x$ , respectively. So, to complete our analysis we need to give the asymptotic representations of  $K_{\alpha,\beta}^\theta(x)$  in the case (ii)  $\{\alpha < \beta\}$  as  $x \rightarrow 0^+$  and in the case (iii)  $\{\alpha > \beta\}$  as  $x \rightarrow +\infty$ . To this purpose we shall adapt to our analysis the results contained in the fundamental paper by Braaksma [2] on the asymptotic expansions for certain Mellin-Barnes integrals. We shall limit ourselves to those cases (not already treated in Sect. 4) where the Green function is expected to be non-negative. As a consequence, for the case (ii) we consider  $0 < \alpha < \beta < 1$ , whereas for the case (iii) we consider  $0 < \beta < \alpha < 2$  with  $\beta \neq 1$ .

For the case (ii)  $\{0 < \alpha < \beta < 1, |\theta| < \alpha\}$  the asymptotic representation as  $x \rightarrow 0^+$  turns out to be given by the convergent representation of the case (iii), namely by the two series in (7.9) or by three series in (7.10)-(7.17). We agree to write (for economy)

$$K_{\alpha,\beta}^\theta(x) \sim \begin{cases} (7.9), & \alpha \neq \frac{m+1}{k+1}, \\ (7.10) - (7.17), & \alpha = \frac{m+1}{k+1}, \end{cases} \quad x \rightarrow 0^+. \quad (7.18)$$

In the extremal case  $\theta = -\alpha$  only the first series in (7.18) [namely in (7.9) or (7.10)] survives, so we obtain the asymptotic representation

$$K_{\alpha,\beta}^\theta(x) \sim \frac{1}{x} \sum_{k=1}^{\infty} \frac{(-x^\alpha)^{k-1}}{\Gamma(\alpha k) \Gamma(1 - \beta k)}, \quad x \rightarrow 0^+. \quad (7.19)$$

We note that in the limiting case  $0 < \alpha < \beta = 1$  all the coefficients of the powers series in (7.19) are vanishing in view of the exponential decay just expected for the unilateral stable distribution  $L_\alpha^{-\alpha}(x)$  as  $x \rightarrow 0^+$ .

For the case (iii)  $\{0 < \beta < \alpha\}$  we distinguish the sub-cases  $0 < \alpha \leq 1$ , where  $|\theta| \leq \alpha$ , and  $1 < \alpha < 2$ , where  $|\theta| \leq 2 - \alpha$ . In both sub-cases, when  $\theta$

does not assume extremal values, the asymptotic representation as  $x \rightarrow +\infty$  turns out to be given by the convergent representation of the case (ii), namely

$$K_{\alpha,\beta}^{\theta}(x) \sim \frac{1}{\pi x} \sum_{n=1}^{\infty} \frac{\Gamma(1+\alpha n)}{\Gamma(1+\beta n)} \sin \left[ \frac{n\pi}{2}(\theta - \alpha) \right] (-x^{-\alpha})^n, \quad x \rightarrow +\infty. \quad (7.20)$$

As for the extremal cases, we have as follows:

If  $0 < \alpha \leq 1$ , the representation (7.20) is still valid for  $\theta = -\alpha$ , whereas is trivially zero for  $\theta = +\alpha$ , as expected. If  $1 < \alpha < 2$ , the representation (7.20) is still valid for  $\theta = 2 - \alpha$ , whereas for  $\theta = \alpha - 2$  we have:

$$K_{\alpha,\beta}^{\theta}(x) \sim A x^a e^{-bx^c}, \quad x \rightarrow +\infty, \quad (7.21)$$

where

$$A = \left\{ 2\pi(\alpha - \beta) \alpha^{\beta/(\alpha-\beta)} \beta^{(\alpha-2\beta)/(\alpha-\beta)} \right\}^{-1/2}, \quad (7.22)$$

$$a = \frac{2\beta - \alpha}{2(\alpha - \beta)}, \quad b = (\alpha - \beta) \alpha^{-\alpha/(\alpha-\beta)} \beta^{\beta/(\alpha-\beta)}, \quad c = \frac{\alpha}{\alpha - \beta}.$$

We note that the above asymptotic representation (7.21)-(7.22) is still valid for  $\beta = 1$ , when it reduces to that for the density  $L_{\alpha}^{\alpha-2}(x)$ , see (4.17), and for  $\alpha = 2$  (so  $\theta = 0$ ), when it reduces to that for the density  $K_{2,\beta}^0(x) = \frac{1}{2} M_{\beta/2}(|x|)$ , see (4.29)-(4.30).

## 8. Conclusive discussion and plots

We can conclude with a discussion about some general features occurring in the Cauchy problem of our space-time fractional diffusion equation (1.1)-(1.2). A first general feature concerns the scaling property of the Green function which allows us to express it in terms of a function of a single variable, the *reduced Green function*  $K_{\alpha,\beta}^{\theta}(x)$ , see (1.3) or (3.8). In this paper we have focused our attention to derive a computational form for  $K_{\alpha,\beta}^{\theta}(x)$  in all of  $\mathbb{R}$ , taking into account the symmetry relation (3.18). The relevant particular cases of space-fractional ( $\{0 < \alpha \leq 2, \beta = 1\}$ ), time-fractional ( $\{\alpha = 2, 0 < \beta \leq 2\}$ ) and neutral-fractional ( $\{0 < \alpha = \beta \leq 2\}$ ) diffusion have been summarized in Sect. 4 where the interpretation of the corresponding Green function as a probability density has been pointed out.

For computational aims the representation of  $K_{\alpha,\beta}^\theta(x)$  through the Mellin-Barnes integral, see (6.6), was found useful. Incidentally, this representation has enabled us to extend the probability interpretation of the reduced Green function to the ranges  $\{0 < \alpha \leq 2\} \cap \{0 < \beta \leq 1\}$  and  $\{1 < \beta \leq \alpha \leq 2\}$ .

More precisely, to compute the function  $K_{\alpha,\beta}^\theta(x)$  we used the series expansions (7.8)-(7.17) and asymptotic expansions (7.18)-(7.22) which were derived from the representation (6.6).

At the end of the section we shall exhibit some plots of the reduced Green function for some "characteristic" values of the parameters  $\alpha$ ,  $\beta$ , and  $\theta$ . All the plots were drawn by using the MATLAB system for the values of the independent variable  $x$  in the range  $|x| \leq 5$ . To give the reader a better impression about the behaviour of the tails the logarithmic scale was adopted.

After long discussions we chose from the uncountable set of possible plots of the function  $K_{\alpha,\beta}^\theta(x)$  the plots having, in our opinion, the greatest interest, in the framework of the probability interpretation.

I. For the Green function of the *space-fractional diffusion* equation ( $\beta = 1$ ), we present plots for the cases:

Fig. 3 :  $\alpha = 0.50$  with a)  $\theta = 0$  and b)  $\theta = -0.50$  ;

Fig. 4 :  $\alpha = 1$  with a)  $\theta = 0$  and b)  $\theta = -0.99$  ;

Fig. 5 :  $\alpha = 1.50$  with a)  $\theta = 0$  and b)  $\theta = -0.50$  .

Note that for  $\alpha = 1$  we have chosen a nearly extremal case to show a density approaching to the delta function  $\delta(x - 1)$ , see (4.10).

II. The fundamental solution of the *time-fractional diffusion* equation ( $\alpha = 2$ ) was plotted in the following cases:

Fig. 6 : a)  $\beta = 0.25$  and b)  $\beta = 0.50$  ;

Fig. 7 : a)  $\beta = 0.75$  and b)  $\beta = 1.25$  ;

Fig. 8 : a)  $\beta = 1.50$  and b)  $\beta = 1.75$  .

III. For the fundamental solution of the *space-time-fractional diffusion* equation, we exhibit the following plots:

Fig. 9 :  $\alpha = 0.50$ ,  $\beta = 0.50$  with a)  $\theta = 0$  and b)  $\theta = -0.50$  ;

Fig. 10 :  $\alpha = 1.50$ ,  $\beta = 1.50$  with a)  $\theta = 0$  and b)  $\theta = -0.49$  ;

Fig. 11 :  $\alpha = 0.25$ ,  $\beta = 0.50$  with a)  $\theta = 0$  and b)  $\theta = -0.25$  ;

Fig. 12 :  $\alpha = 0.75$ ,  $\beta = 0.50$  with a)  $\theta = 0$  and b)  $\theta = -0.75$  ;

Fig. 13 :  $\alpha = 1.50$ ,  $\beta = 0.50$  with a)  $\theta = 0$  and b)  $\theta = -0.50$  ;

Fig. 14 :  $\alpha = 1.50$ ,  $\beta = 1.25$  with a)  $\theta = 0$  and b)  $\theta = -0.50$  .

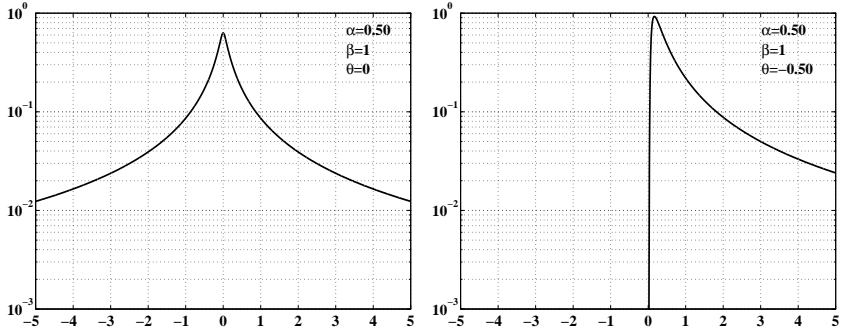


Fig. 3

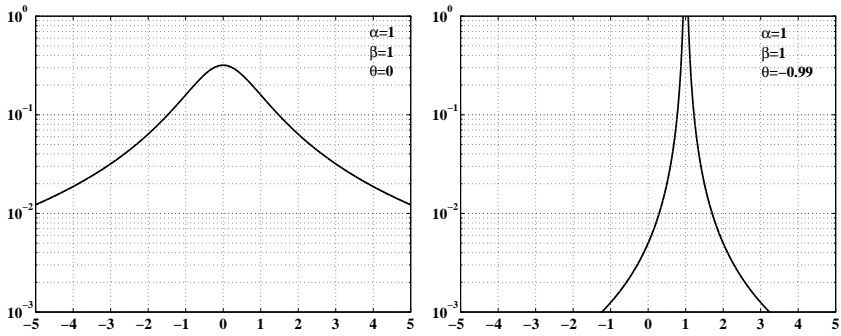


Fig. 4

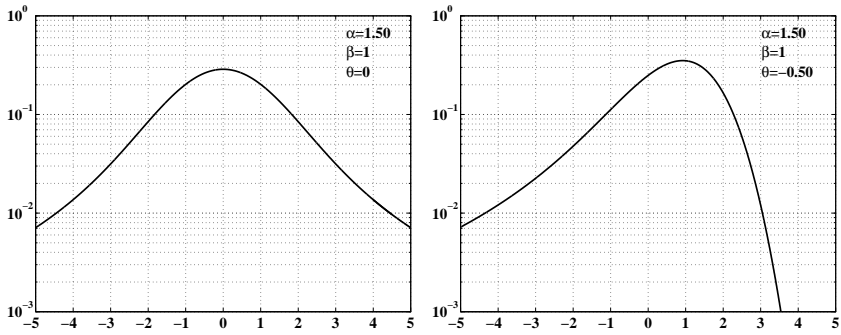


Fig. 5

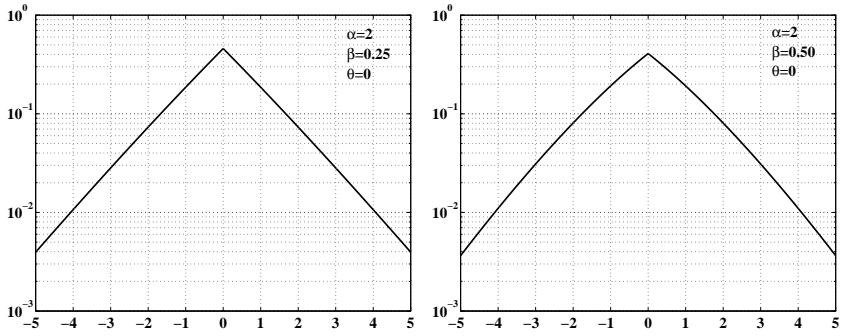


Fig. 6

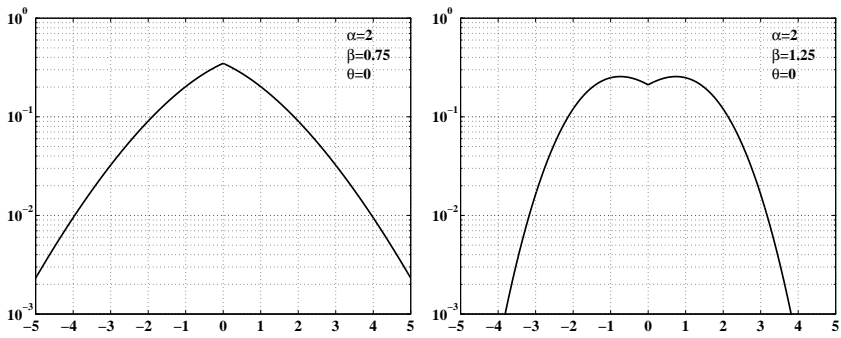


Fig. 7

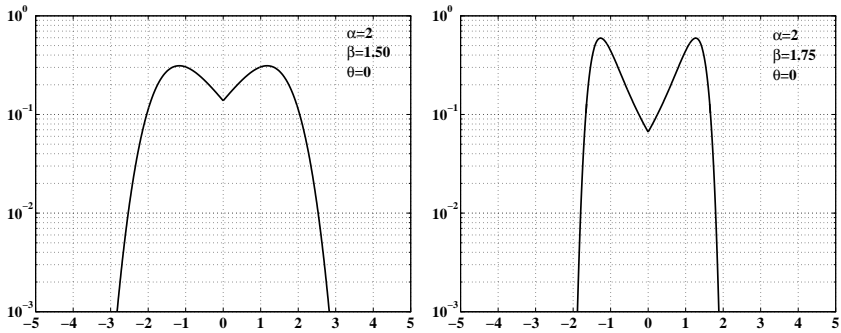


Fig. 8

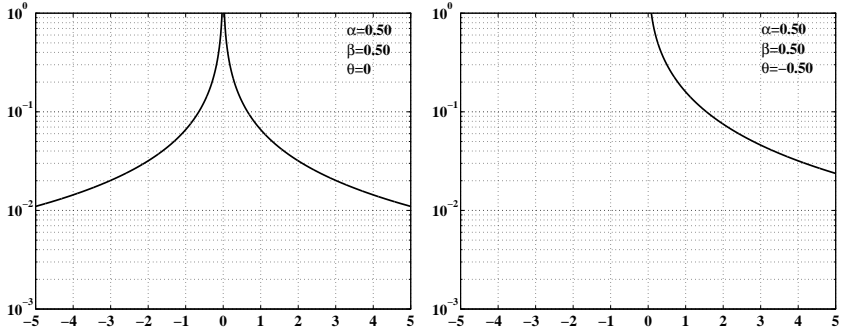


Fig. 9

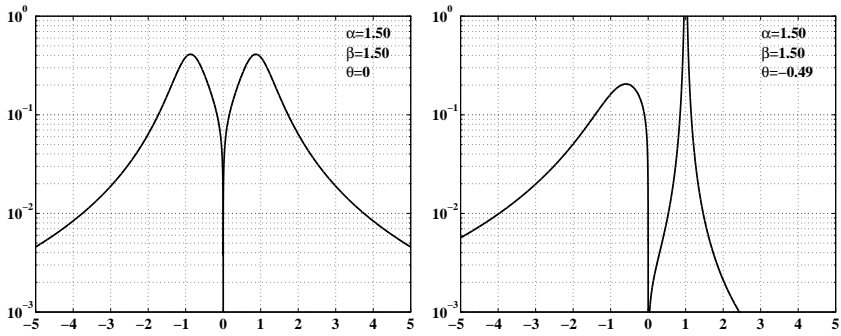


Fig. 10

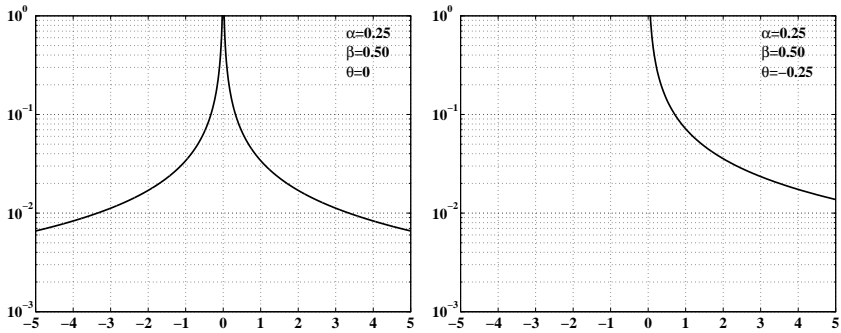


Fig. 11



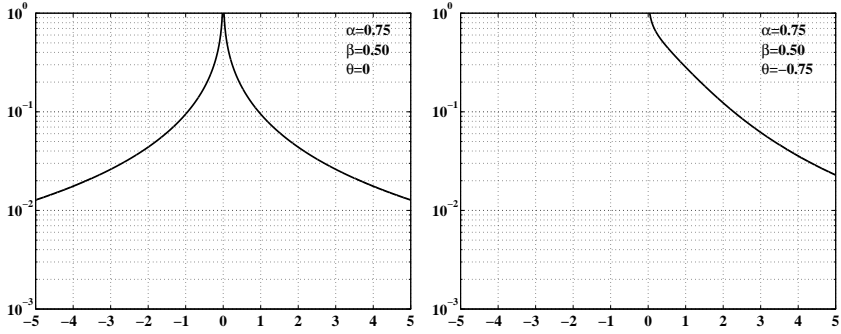


Fig. 12

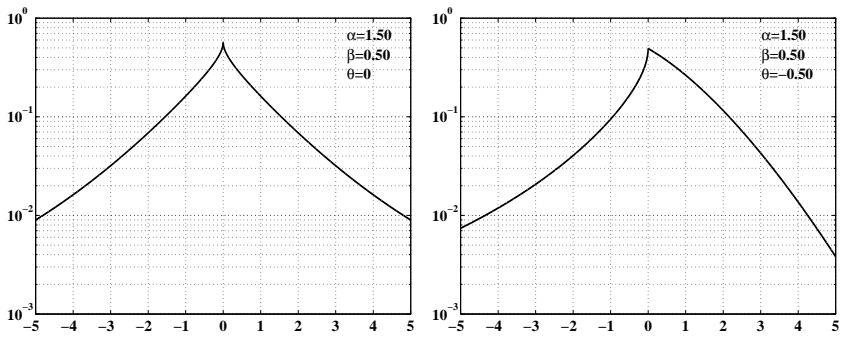


Fig. 13

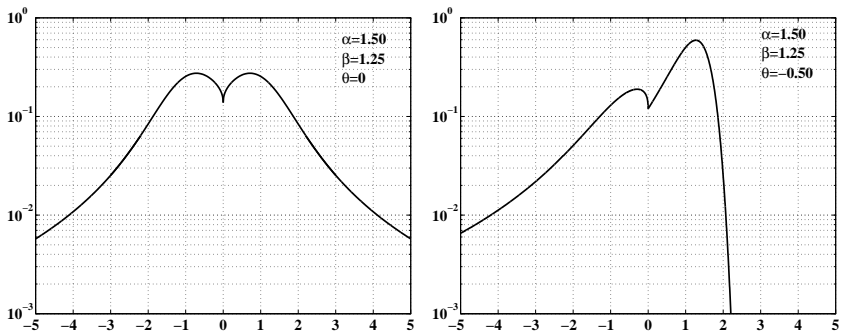


Fig. 14

## Acknowledgements

The authors are very grateful to Prof. Dr. R. Gorenflo for the inspiring discussions and the helpful comments.

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